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FOUNDATIONS

Rose, Gene F. Propositional calculus and realizability. Trans. Amer. Math. Soc. 75, 1–19 (1953).

The first part of this paper contains a detailed proof of Jaśkowski's theorem about the infinite characteristic matrix for the intuitionistic propositional calculus, for which until now only very incomplete indications were available in the literature. Moreover, by successive applications of Jaśkowski's Γ-operation on the matrix L_0 of one element, a sequence of matrices $\{L_n\}$ is defined. The second part contains new results about the theory of realizability. This notion was introduced for number-theoretic formulas (n-formulas) by Kleene [J. Symbolic Logic 10, 109–124 (1945); cf. "Introduction to metamathematics," Van Nostrand, New York, 1952, §82; these Rev. 7, 406; 14, 525]. A formula of the propositional calculus (p-formula) is said to be realizable if every n-formula obtained from it by substituting n-formulas for its p-variables is realizable. Let $\mathcal{H}$ be the set of provable formulas in the intuitionist propositional calculus, $\mathcal{P}$ the set of realizable p-formulas, $\mathcal{L}$ the set of p-formulas which fulfill every L_n, $\mathcal{C}$ the set of provable formulas in the classical propositional calculus. Nelson [Trans. Amer. Math. Soc. 61, 307–368 (1947); these Rev. 10, 3] proved that every provable formula in intuitionistic number theory is realizable; it follows that $\mathcal{H}\subseteq\mathcal{P}$. Here an example is given of a p-formula in $\mathcal{P}$ but not in $\mathcal{H}$. Also it is proved that $\mathcal{P}\subseteq\mathcal{L}$. In each of these two proofs a non-intuitionist application of the law of excluded middle occurs. As $(a\supset b)\vee(b\supset a)$ is in $\mathcal{L}$, but not in $\mathcal{P}$, and $a\vee\neg a$ is in $\mathcal{C}$ but does not fulfil L_2, we have (classically) the following chain of proper set-inclusions: $\mathcal{H}\subset\mathcal{P}\subset\mathcal{L}\subset\mathcal{C}$. *A. Heyting* (Amsterdam).

Meredith, Carew A. Single axioms for the systems (C, N), (C, O) and (A, N) of the two-valued propositional calculus. J. Computing Systems 1, 155–164 (1953).

The author considers three forms of the 2-valued propositional calculus in which the primitives are implication and negation, implication and the logical constant O, and alternation and negation respectively. Taking, in each case, substitution and modus ponens as primitive rules of procedure he shows that the formulae

$$CCCCCpqCNrNsrtCCtpCsp,\quad CCCCCpqCrOstCCtpCrp,$$
$$ANANANpqArAstANANspArAtp$$

can be taken as single axioms for the respective systems. He states, without proof, an alternative axiom of 19 letters for the (C, O) system and two alternative axioms of 24 letters for the (A, N) system. He conjectures that his axioms for the (C, N) and (C, O) systems are the shortest possible and shows that a single axiom for the (A, N) system must contain at least five N's. *A. Rose* (Nottingham).

Meredith, Carew A. A single axiom of positive logic. J. Computing Systems 1, 169–170 (1953).

The author deduces the Łukasiewicz axioms $CpCqp$, $CCpCqrCCpqCpr$ from the single axiom $CCCpqrCsCCqCrtCqt$. He raises the question whether this is the shortest axiom of the system but does not discuss it in detail. *A. Rose.*

Stanley, Robert. An extended procedure in quantificational logic. J. Symbolic Logic 18, 97–104 (1953).

The author describes a certain (rather involved) procedure to be applied to schemata (closed expressions) of quantification theory. In certain cases, this procedure, if repeated sufficiently often, comes to a stop, and then the schema under consideration can be stated to be valid. The reviewer is even satisfied that every valid schema can be detected by repeated application of the procedure. Of course, as by Church's theorem there can be no decision method for validity in quantification theory, nothing definite can be said in advance as to the number of repetitions required for testing a given schema. The author has successfully applied the procedure to a number of schemata known to be valid, and he shows that it entails the known general decision method for monadic validity. No results are claimed concerning other solvable cases of the decision problem, but presumably a more detailed study of the procedure might result in new solutions for special cases. *E. W. Beth.*

Quine, W. V. On ω-inconsistency and a so-called axiom of infinity. J. Symbolic Logic 18, 119–124 (1953).

The author stresses that the notion of ω-consistency depends upon the definition of "natural number" which is adopted. Let S be a formal system which contains quantification theory and for which at least a partial interpretation is known such that the natural numbers are expressible in S in the following sense: there is a standard method whereby, for each natural number i and each sentence φ of S having 'x' as sole free variable, a statement φ_i of S can be formed which is true if and only if φ is satisfied by i as value of 'x'. φ is said to be numerically general if φ_i is a theorem of S for each i. Let, in addition, some numerically general sentence ψ in 'x' be interpreted as "x is a natural number"; then S is ω-inconsistent if there is a numerically general χ such that $\ulcorner(\exists x)(\psi\bar{\chi})\urcorner$ is a theorem of S. In this case, $\ulcorner\psi\chi\urcorner$ will be a better translation of "x is a natural number". If this process does not terminate, the author proposes to call S numerically insegregative (n.i.). Whether Quine's system of "New foundations" (NF) [Amer. Math. Monthly 44, 70–80 (1937)] is n.i., is an open question, though Rosser and Wang [J. Symbolic Logic 15, 113–129 (1950); these Rev. 12, 384] proved that under Frege's definition of "x is a natural number" NF possesses no model which preserves well-ordering in the theory of ordinal numbers. *A. Heyting* (Amsterdam).

★Łukasiewicz, Jan. Sur la formalisation des théories mathématiques. Les méthodes formelles en axiomatique. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 26, Paris, 1950, pp. 11–19; discussion, pp. 19–21. Centre National de la Recherche Scientifique, Paris, 1953.

The author discusses the mutual relationship of various fundamental principles of the elementary theory of numbers, such as weak induction, strong induction, principle of

descent, the well-ordering principle ("principle of the least number"). He gives completely formalised proofs of the equivalence of the three last-mentioned principles. The author uses a formal system which is expressed by means of his bracketless notation and includes the variable propositional functor δ [cf. J. Łukasiewicz, Proc. Roy. Irish Acad. Sect. A. **54**, 25–35 (1951); these Rev. **13**, 3].

In the discussion that follows, H. B. Curry compares the author's bracketless notation with other systems. He concludes that in some cases clarity demands the provisional insertion of brackets or dots even in the Łukasiewicz notation.

A. Robinson (Toronto, Ont.).

Łukasiewicz, Jan. A system of modal logic. J. Computing Systems **1**, 111–149 (1953).

In the first part of this paper Łukasiewicz develops a "basic modal logic". By this is understood a system containing the expressions "Δp" and "Γp", denoting "It is possible that p" and "It is necessary that p" respectively, and such that the system has the following eight properties, the symbols "$\vdash$" and "$\dashv$" denoting assertion and rejection respectively: $\vdash Cp\Delta p$, $\dashv C\Delta pp$, $\dashv \Delta p$, $\vdash C\Gamma pp$, $\dashv Cp\Gamma p$, $\dashv N\Gamma p$, $\vdash E\Delta pN\Gamma Np$, $\vdash E\Gamma pN\Delta Np$. A system is called "modal logic" if and only if it includes the basic modal logic as a part.

Basic modal logic is formalised by means of the axioms $\vdash Cp\Delta p$, $\dashv C\Delta pp$, $\dashv \Delta p$, $\vdash E\Delta p\Delta NNp$ together with the axioms of the classical propositional calculus, necessity being defined by $\vdash E\Gamma pN\Delta Np$. The rules of procedure are substitution and modus ponens together with corresponding rules for rejection. Independence proofs are given for the four Δ-axioms. Łukasiewicz then discusses Aristotle's theorems of propositional modal logic and shows that certain related formulae of the present system cannot be deduced from the axioms.

In the second part of the paper the Ł-modal system is developed. This includes the general principle of extentionality, $CEpqC\delta p\delta q$, where "δ" denotes a variable functor. The Ł-system is formalised by means of the axioms $\vdash C\delta pC\delta Np\delta q$, $\vdash Cp\Delta p$, $\dashv C\Delta pp$, $\dashv \Delta p$ together with the above rules of procedure, the substitution rule being taken in a generalised form. The axioms are shown to be independent. It is then shown that the matrix given below on the left is adequate for the system.

C	1	2	3	4	N	Δ		∇
1*	1	2	3	4	4	1	1	1
2	1	1	3	3	3	1	2	2
3	1	2	1	2	2	3	3	1
4	1	1	1	1	1	3	4	2

It is also shown that if the function ∇ has the truth-table given above on the right then if ∇, Δ do not occur in P, Q respectively and Q is obtained from P by replacing Δ throughout by ∇ then P satisfies the matrices if and only if Q does. This replacement rule cannot be extended to formulae containing both Δ and ∇. $\Delta\nabla p$ satisfies the matrices but $\Delta\Delta p$ does not.

Some provable formulae of Ł-modal logic are then discussed, it being shown that there exist formulae P such that $\dashv P$ and $\vdash \Delta P$. The paper concludes with a discussion of some controversial problems.

A. Rose (Nottingham).

McKinsey, J. C. C. Systems of modal logic which are not unreasonable in the sense of Halldén. J. Symbolic Logic **18**, 109–113 (1953).

A system is said to be unreasonable (in the sense of Halldén) if there exists a pair of formulae α and β such that: (i) α and β contain no variable in common; (ii) neither α nor β is provable; (iii) $\alpha \vee \beta$ is provable. It is first shown that if there exists a matrix $\langle K, D, X, -, {}^*\rangle$ which is a characteristic matrix for a system S and such that D is an additive prime ideal of the Boolean algebra $\langle K, X, -\rangle$ then S is not unreasonable in the sense of Halldén. From this McKinsey deduces that the Lewis system S4 is not unreasonable in the sense of Halldén.

It is then shown that no extension of S5 which is quasi-normal in the sense of Scroggs [same J. **16**, 112–120 (1951); these Rev. **13**, 97] is unreasonable in the sense of Halldén. In conclusion McKinsey shows that there are unreasonable systems between S4 and S5.

A. Rose (Nottingham).

***Johansson, Ingebrigt. Sur le concept de "le" (ou "ce qui") dans le calcul affirmatif et dans les calculs intuitionnistes.** Les méthodes formelles en axiomatique. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 26, Paris, 1950, pp. 65–72; discussion, p. 72. Centre National de la Recherche Scientifique, Paris, 1953.

The author indicates how descriptions can be introduced in a theory T which is formalized over the positive logic, the minimal calculus or the intuitionistic logic. It is supposed that T contains the equality sign. If p is any term of T and x a variable, the substitution operator p_x is defined by $p_xf(\cdots x\cdots)\overline{\mathrm{D}}f(\cdots p\cdots)$. Eight equivalence formulas S_1-S_8 for substitution operators are easily derived. Let A be the conjunction of the axioms of T. If $u(x)$ is a propositional function such that the formula

$$A \supset (Ex)u(x) \;\&\; (x)(y)(u(x) \;\&\; u(y) \supset x=y)$$

is derivable, then a description operator $(\iota_xu(x))_y$ is introduced and the axiom $(\iota_xu(x))_yf(y) \supset (y)(u(y) \supset f(y))$ is added to T. It is shown that description operators satisfy S_1-S_8, so that in this respect $\iota_xu(x)$ may be treated as a term.

A. Heyting (Amsterdam).

Büchi, J. Richard. Investigation of the equivalence of the axiom of choice and Zorn's lemma from the viewpoint of the hierarchy of types. J. Symbolic Logic **18**, 125–135 (1953).

Letters with subscripts α, 0α, $\alpha(0\alpha)$, $0(\alpha\alpha)$ denote resp. sets of type α, properties of such sets, functions whose arguments are such properties and whose values are sets of type α, relations between such sets. Zermelo's axiom for sets of type α: $\exists h_{\alpha(0\alpha)}(a_{0\alpha})[ax \supset a(ha)]$ or ZA^α. Zorn's lemma for sets of type α: $(r_{0(\alpha\alpha)})[Pr \wedge Wr] \supset (\exists x_\alpha)(u_\alpha)(rxu \supset rux)$ or $(r_{0(\alpha\alpha)})ZL^\alpha r$, where Pr is the (natural) formalization of 'r quasi-orders the type α' and Wr of 'every r-chain has an r-upper bound'.

Analysis of an argument due to the author yields a proof of $ZA^\alpha \supset (r_{0(\alpha\alpha)})ZL^\alpha r$ in the logical functional calculus whose bound variables are restricted to the types α, 0α, $\alpha(0\alpha)$, $0(\alpha\alpha)$ (l.f.c. α) [see Church, same J. **5**, 56–68 (1940); these Rev. **1**, 321]. The (natural) formalization of an argument of Birkhoff's Lattice theory [Amer. Math. Soc. Colloq. Publ., v. 25, New York, 1940; these Rev. **1**, 325] yields a proof in l.f.c. α of $ZL^{\alpha(0\alpha)}R \supset ZA^\alpha$ where R is a suitable relation between functions of type $\alpha(0\alpha)$. The author conjectures

that there are no types α, β such that $ZA^{\alpha}\equiv(r_{0(\beta\beta)})ZL^{\beta}r$ in a logical functional calculus restricted to a fixed number of types.

The informal equivalence may be restated thus: ZA^{α} for all α is equivalent to: $(r_{0(\beta\beta)})ZL^{\beta}r$ for all β. [Reviewer's note: a proof of the author's conjecture would show that a straightforward interpretation of classical set theory by a set theory with types is impossible, not only for a constructive (ramified) but also for the simple hierarchy of types.]

G. Kreisel (Reading).

Hirsch, Guy. Théorie et expérience en mathématiques. Dialectica 6, 311–326 (1952).

Partant de certaines considérations de F. Gonseth [La géométrie et le problème de l'espace, IV, Griffon, Neuchâtel, 1949] concernant la notion du schéma, l'auteur fait remarquer que tout schéma a son domaine d'efficacité et que les découvertes (antinomies, contre-exemples) qui mettent en évidence les limites de ce domaine sont, par rapport au schéma, ce que, dans les sciences de la nature, est l'expérience par rapport à une théorie. Ces observations conduisent l'auteur à poser le problème de développer une dialectique de l'erreur qui réunirait les différentes techniques auxquelles recourt, de façon plus ou moins consciente, le mathématicien lorsqu'il se trouve en présence d'une discordance entre théorie et expérience. Il ne montre pas en quoi cette dialectique de l'érreur différerait de la méthodologie des mathématiques telle qu'elle existe.

E. W. Beth.

ALGEBRA

Yamamoto, Koichi. Symbolic methods in the problem of three-line Latin rectangles. J. Math. Soc. Japan 5, 13–23 (1953).

The methods previously employed by the author in the study of three-line Latin rectangles [Sûgaku 2, 159–162 (1949); J. Math. Soc. Japan 1, 226–241 (1950); these Rev. 12, 494] are given an alternative development which results in the following formula for the number of such rectangles

$$f(3,n)=(n!)^2\sum_{k=0}^{n}H_k[(2-3F)/2\sqrt{F}]\frac{F^{3k/2}}{k!(n)_k}\sigma_n.$$

In this, $H_k(x)$ is an Hermite polynomial:

$$H_0=1,\quad H_{k+1}=2xH_k-2kH_{k-1},$$

F is a shift operator: $F\sigma_n=\sigma_{n-1}$, $(n)_k=n!/(n-k)!$, and σ_n is the truncated exponential series $\sum_0^n(-3)^n/n!$. This leads immediately to the author's asymptotic series (in the second paper cited above) and to the reviewer's recurrence relation [Amer. Math. Monthly 59, 159–162 (1952); these Rev. 13, 813].

J. Riordan (New York, N. Y.).

Bose, R. C., Shrikhande, S. S., and Bhattacharya, K. N. On the construction of group divisible incomplete block designs. Ann. Math. Statistics 24, 167–195 (1953).

The authors describe a variety of methods, most of them new, for the construction of group divisible incomplete block designs (G.D.). A short enumeration of these methods follows: (1) Replacement of each treatment of a balanced incomplete block design (B.I.B.) by a group of n treatments. This gives all singular G.D. as shown in a previous paper [R. C. Bose and W. S. Connor, same Ann. 23, 367–383 (1952); these Rev. 14, 124]. (2) Omission of one treatment and all blocks containing it from a B.I.B. with $\lambda=1$. This gives G.D. with $\lambda_1=0$, $\lambda_2=1$, semi-regular if the B.I.B. is symmetric, regular otherwise. (3) Two methods which are modifications of the method of symmetrically repeated differences of the R. C. Bose. (4) Derivation of G.D. with $\lambda_1=0$ from orthogonal arrays of strength two. (5) Several methods by which new G.D. may be constructed from designs already known. All these methods are well illustrated by examples and many solutions within the range $v\leqq10$, $r\leqq10$, $k\leqq10$ are given explicitly.

H. B. Mann.

Carlitz, L. Permutations in a finite field. Proc. Amer. Math. Soc. 4, 538 (1953).

A polynomial $f(x)$ with coefficients in $GF(q)$ is called a permutation polynomial if the numbers $f(\alpha)$, where $\alpha\in GF(q)$, are a permutation of the α's. The author proves very simply that all permutation polynomials can be generated by means of the special types $\alpha x+\beta$, x^{q-2} ($\alpha,\beta\in GF(q)$, $\alpha\neq0$).

A. L. Whiteman (Princeton, N. J.).

Carlitz, L. Note on some formulas of Rodeja F. Proc. Amer. Math. Soc. 4, 528–529 (1953).

Further simplification of the proof of an arithmetic identity used by the reviewer [Trans. Amer. Math. Soc. 63, 175–192 (1948); these Rev. 9, 421].

A. W. Goodman (Lexington, Ky.).

Mikusiński, J. G.-. Sur un déterminant. Ann. Soc. Polon. Math. 25 (1952), 27–29 (1953).

The determinant considered generalizes that of Vandermonde, and the product expansion obtained is a generalization of the well-known expression for the latter.

L. M. Blumenthal (Columbia, Mo.).

Seidel, W. Note on a persymmetric determinant. Quart. J. Math., Oxford Ser. (2) 4, 150–151 (1953).

The author shows that a result of Burchnall's [same J. (2) 3, 151–157 (1952); these Rev. 14, 44] may be deduced from a generating function given by Brafman [Proc. Amer. Math. Soc. 2, 942–949 (1951); these Rev. 13, 649] by means of the theory of continued fractions.

A. Erdélyi.

Perfect, Hazel. Methods of constructing certain stochastic matrices. Duke Math. J. 20, 395–404 (1953).

Continuing earlier work [Proc. Cambridge Philos. Soc. 48, 271–276 (1952); these Rev. 13, 760] the author shows that PDP^{-1} is a positive stochastic $(n+1)\times(n+1)$ matrix when D is the diagonal matrix $1, \lambda_1, \cdots, \lambda_n$ $(0\leqq\lambda_i<1)$ and P is the matrix (p_{ik}) with $p_{i,n-k+2}=1$, $i\leqq k$, $p_{i,n-i+3}=-1$ and zeros elsewhere. Apart from this, two sufficient conditions are obtained for $1, \lambda_i$ $(i=1,\cdots,n)$, $|\lambda_i|\leqq1$, to be the characteristic roots of a stochastic matrix. One of these is as follows: There is a possible subdivision of the set 1, $\lambda_1,\cdots,\lambda_n$ into subsets each of which contains at least one positive number and such that the sum of the moduli of the negative numbers in the subset does not exceed the greatest positive number in it. The second condition, too involved to be stated here, is shown not to be contained in the first one. In both cases the sufficiency is established by means of a theorem of A. Brauer [Duke Math. J. 19, 75–91 (1952), theorem 27; these Rev. 13, 813] for which a new proof is presented. [Brauer's theorem states that the $n\times n$ matrix A with characteristic root ω and $x=(x_1,\cdots,x_n)$ as

a corresponding vector has its other characteristic roots coinciding with $B=(a_{ik}-x_ih_k)$, where h_i are arbitrary. An alternative proof can be found by observing that the matrix $C=(x_ih_k)$ has as characteristic roots $\sum x_ih_i$ and $n-1$ zeros. It is further easy to see that the matrices A and C can be transformed to triangular form by the same similarity. For, let SAS^{-1} be triangular with ω in the upper left corner. Then x goes over into Sx which is the vector $(1, 0, \cdots, 0)$. Hence SCS^{-1} is triangular with $n-1$ zeros in the main diagonal. From this the assertion follows.]

O. Taussky-Todd (Washington, D. C.).

Goldhaber, J. K., and Whaples, G. On some matrix theorems of Frobenius and McCoy. Canadian J. Math. **5**, 332–335 (1953).

Earlier results of Goldhaber [same J. **4**, 31–42 (1952); these Rev. **13**, 619] as well as a theorem of McCoy [Bull. Amer. Math. Soc. **42**, 592–600 (1936)] are proved under more general assumptions than before. The following definitions are used. Let $\lambda_i(A)$ be the characteristic roots of the matrix A. An algebra $\mathfrak{A}$ of $n\times n$ matrices A_i which contains the identity matrix, over a field k, is said to have property F if for every polynomial $f(x_1, \cdots, x_m)$ and every finite subset $A_1, \cdots, A_m$ of $\mathfrak{A}$ there is an ordering such that $\lambda_i f(A_1, \cdots, A_m)=f(\lambda_i(A_1), \cdots, \lambda_i(A_m))$. Property M means that $\mathfrak{A}$ is commutative with respect to its radical. Property P holds if for every A, B there is an ordering such that $\lambda_i(A+B)=\lambda_i(A)+\lambda_i(B)$. Property P′ holds if the sum of two nilpotent elements of $\mathfrak{A}$ is nilpotent. It is proved that for arbitrary k (not necessarily algebraically closed) and $\mathfrak{A}$ the properties F, M and P are equivalent. If k is quasi-algebraically closed (i.e., not the centre of any non-commutative division algebra) then F, M, P and P′ are equivalent. The following lemma which was previously only proved for characteristic zero is now proved generally. If $A\in\mathfrak{A}$ and N belongs to the radical of $\mathfrak{A}$ then A and $A+N$ have the same characteristic functions. [The names of some of the properties used do not coincide with those used by previous authors. It is interesting to observe that properties P and F (in the sense of this paper) if assumed for a pencil of matrices instead of a whole algebra do not coincide [T. S. Motzkin and O. Taussky, Trans. Amer. Math. Soc. **73**, 108–114 (1952); these Rev. **14**, 236]. An algebra with property F over a not algebraically closed field had been studied previously in a special case [O. Taussky and J. Todd, J. London Math. Soc. **17**, 146–151 (1942); these Rev. **4**, 185].] *O. Taussky-Todd* (Washington, D. C.).

Bartsch, Helmut. Ein Einschliessungssatz für die charakteristischen Zahlen allgemeiner Matrizen-Eigenwertaufgaben. Arch. Math. **4**, 133–136 (1953).

H. Wielandt [Arch. Math. **1**, 348–352 (1949); these Rev. **11**, 4] and A. G. Walker and J. D. Weston [J. London Math. Soc. **24**, 28–31 (1949); these Rev. **10**, 501] proved the following inclusion theorem for the eigenvalues of a normal $n\times n$ matrix A: Let $(x_1, \cdots, x_n)$ be any column vector. Any circle which contains all the quotients $q_j=(Ax)_j/x_j$ contains at least one eigenvalue of A. A corresponding theorem does not hold for the generalized eigenvalue problem $Ax=\lambda Bx$ when the q_j are replaced by $(Ax)_j/(Bx)_j$, even if A, B are both real, symmetric and positive definite. However, the following theorem can be proved for A, B hermitian and B positive definite: If a circle with radius r contains the q_j, then the circle with the same centre and radius $r(\lambda_{\max}/\lambda_{\min})^{1/2}$ contains at least one generalized eigenvalue where $\lambda_{\max}$, $\lambda_{\min}$ are the largest and smallest eigenvalues of B. *O. Taussky-Todd.*

Abstract Algebra

★Zacher, Giovanni. Sugli emiomorfismi superiori ed inferiori. Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 251–252. Casa Editrice Perrella, Roma, 1953.

The author announces various results on meet-homomorphisms (and their duals) of lattices. A meet-homomorphic image of a finite-dimensional lattice is a lattice, without need to postulate any join operation in the image. Subject to simple limitations, every lattice has as join-homomorphic image any lattice, with no greater number of atoms, in which every element of dimension greater than one covers at least two elements. *P. M. Whitman.*

Thrall, R. M., and Duncan, D. G. Note on free modular lattices. Amer. J. Math. **75**, 627–632 (1953).

The free modular lattice generated by $2+1+1$ has 138 elements. This was found by Takeuchi, thus solving Problem 29 of the reviewer's "Lattice theory" [rev. ed., Amer. Math. Soc. Colloq. Publ., v. 25, New York, 1948; these Rev. **10**, 673]. The authors make a few changes in Takeuchi's formulas. They note that it is still unknown whether or not the free modular lattice generated by $3+1+1$ is infinite, and that the same uncertainty prevails concerning that generated by $(n\circ 2^2)+1$ (i.e., a chain of n "diamond" lattices $D=2^2$, and a single incomparable element).

G. Birkhoff (Cambridge, Mass.).

Blair, Robert L. Ideal lattices and the structure of rings. Trans. Amer. Math. Soc. **75**, 136–153 (1953).

As is well known, the set of all ideals or right ideals of an arbitrary ring A forms a complete modular lattice. The author studies the effect of various restrictions which are imposed on these lattices upon the structure of the ring. The ring A is said to satisfy condition C (resp. C_r) if the lattice of ideals (resp. right ideals) is complemented. It is shown that A satisfies condition C (resp. C_r) if and only if A is isomorphic with a discrete direct sum of simple rings (resp. if and only if A is atomic, i.e., it coincides with its socle). If A is semi-simple, condition C_r implies condition C. The structure space of a ring A [cf. N. Jacobson, Proc. Nat. Acad. Sci. U. S. A. **31**, 333–338 (1945); these Rev. **7**, 110] satisfying condition C (resp. C_r) is discrete, and it is compact if and only if in addition there is an element in A which is in no primitive ideal of A. One of the consequences is: The existence of an identity element together with condition C_r is equivalent to semi-simplicity together with the minimal condition for right ideals. The author turns further to the case where the lattice of ideals (resp. right ideals) is distributive (in short: A satisfies condition D, resp. condition D_r). He calls an ideal (right ideal) I strongly irreducible in case $B\cap C\subseteq I$ implies that $B\subseteq I$ or $C\subseteq I$ for any ideals (resp. right ideals) B and C, and shows that A satisfies condition D (resp. D_r) if and only if each ideal (right ideal) of A is the intersection of all strongly irreducible ideals (right ideals) which contain it. He also proves that condition D_r and semi-simplicity imply that A is isomorphic with a subdirect sum of division rings. A ring A is called f-regular if for any $a\in A$ we have $(a)=(a)^2$, where (a) is the ideal generated by a. The author observes that any f-regular

ring satisfies condition D, and that f-regularity is equivalent with any of the following two conditions: (1) $B\cdot C=B\cap C$ for ideals B and C in A; (2) for each ideal B of A, $A-B$ contains no nonzero nilpotent ideals. A consequence of this equivalence is the following result: A ring A is f-regular if and only if each ideal of A is the intersection of all prime ideals which contain it. Since regularity (and biregularity) imply f-regularity, these results generalize two theorems due to M. C. Waddel [Duke Math. J. **19**, 623–627 (1952), Theorems 2 and 3; these Rev. **14**, 348]. Finally we note the following result: The lattice of right ideals in a semi-simple ring A satisfies condition C_r as well as condition D_r (i.e., this lattice forms a Boolean algebra) if and only if A is a discrete direct sum of division rings. *J. Levitzki.*

Witt, Ernst. Über freie Ringe und ihre Unterringe. Math. Z. **58**, 113–114 (1953).

The celebrated Theorem of Schreier says that every subgroup of a free group is free. It is also true that every subgroup of a free abelian group is free abelian. It is not however true that subrings of free associative rings are free, as may be seen by the example of the subring generated by x^2 and x^3 in the ring generated by x. On the other hand there are non-associative free rings whose subrings are free. These include free distributive rings and free Lie rings. The proof for distributive rings is given here, but not that for Lie rings. *Marshall Hall, Jr.* (Columbus, Ohio).

Ribenboim, Paulo. Modules sur un anneau de Dedekind. Summa Brasil. Math. **3**, 21–36 (1952).

Let A be a Dedekind ring, K its quotient field, $P=\{p\}$ the set of all prime ideals of A, F the set of all functions f on P with values which are integers or $-\infty$, and subject to the condition $f(p)\leqq 0$ for almost all p. If $f\varepsilon F$, associate with it the A-submodule M of K consisting of all x such that $v_p(x)\geqq f(p)$, v_p being the valuation of K determined by p. Conversely, if M is an A-submodule $\neq(o)$ of K, associate with it the function f: $f(p)=\min_{x\varepsilon M}\{v_p(x)\}$. It is proved that these two mappings are inverse and hence establish a 1-1 correspondence between F and the set of all A-submodules $M\neq(o)$ of K. Here M is a fractional ideal if $f(p)\neq-\infty$ for all p and $f(p)\geqq 0$ for almost all p; M is a ring if $f(p)=-\infty$ or $f(p)\geqq 0$ for every p. If the elements of F are partially ordered coordinatewise, then the correspondence is order reversing and converts sum, intersection, product of submodules into inf, sup, sum of elements of F. The module M is discussed in relation to the largest subring B of K such that M is a B-module. *I. S. Cohen.*

***Fuchs, László. On a new type of radical.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 435–443. Akadémiai Kiadó, Budapest, 1952. (Hungarian. Russian and English summaries)

The author observes that the Jacobson radical of a ring often contains elements which are not divisors of zero, unlike classical radicals which even consist of nilpotent elements. He proposes a new radical designed to contain only divisors of zero. Call z a zeroid element if $z+a$ is a left divisor of zero whenever a generates a two-sided ideal consisting entirely of left divisors of zero. The radical Z is to be the union of all ideals consisting of zeroid elements. Examples reveal that Z is incomparable with the Jacobson radical J. For the ring of p-adic integers, $J=(p)$ and $Z=0$. For the ring of polynomials in two variables reduced modulo $I=(x^2, xy)$, we have $J=(x)/I$, $Z=(x, y)/I$. A detailed account in English is promised for a later paper. *I. Kaplansky* (Chicago, Ill.).

Eckmann, B., und Schopf, A. Über injektive Moduln. Arch. Math. **4**, 75–78 (1953).

Let R be a ring with an identity element. An R-module M is an abelian group on which R operates from the left in such a way that the identity element of R acts as the identity operator. M is said to be R-injective if, for every pair of R-modules $B\subset A$, every R-homomorphism of B into M can be extended to an R-homomorphism of A into M. By a very simple and elegant argument it is shown that every R-module can be imbedded in an injective R-module. It is pointed out that this result is equivalent to a theorem of R. Baer [Bull. Amer. Math. Soc. **46**, 800–806 (1940); these Rev. **2**, 126]. In the present form, it is one of the basic tools in the (forthcoming) general cohomology theory of Cartan-Eilenberg.

An R-module M_1 containing M is called an essential extension of M if every non-zero submodule of M_1 has a non-zero intersection with M. An imbedding of M in an injective R-module N can always be extended to an imbedding in N of any essential extension of M. It follows that M has a maximal essential extension, and that the following three conditions on M are equivalent: (a) M is injective; (b) M has no proper essential extensions; (c) M is a direct summand in every extension. Finally, it is concluded that every R-module has a unique minimal injective extension, namely its (unique) maximal essential extension. *G. Hochschild* (Urbana, Ill.).

Dieudonné, Jean. A problem of Hurwitz and Newman. Duke Math. J. **20**, 381–389 (1953).

K being a sfield of characteristic $\neq 2$, E an n-dimensional right vector space over K, σ an automorphism of K and γ an element of K such that $\gamma^\sigma=\gamma$ and $\xi^{\sigma^2}=\gamma^{-1}\xi\gamma$ for $\xi\varepsilon K$, the author considers the problem of determining the maximal number p of semi-linear transformations $u_k(1\leqq k\leqq p)$ of E relative to σ satisfying $u_k^2(x)=x\gamma$ $(x\varepsilon E$, all $k)$, $u_hu_k=-u_ku_h$ $(h\neq k)$. The special cases $K=$ field of complex numbers, $\sigma=$identity, $\gamma=1$, resp. -1, have been solved by A. Hurwitz [Math. Ann. **88**, 1–25 (1922)], resp. M. H. A. Newman [J. London Math. Soc. **7**, 93–99 (1932)]. Using methods he has developed for studying systems of involutive collineations or correlations which are projectively permutable [Summa Brasil. Math. **2**, no. 6, 59–94 (1950); Acta Math. **87**, 175–242 (1952); these Rev. **13**, 531; **14**, 239], the author shows how his problem may be solved, and carries through the details for the following general subcases: (I) -1 is a square in the center of K; (II) K is commutative, $\sigma=$identity, -1 is not a square in K; (III) $\sigma=$identity, $\gamma=-1$. *E. R. Kolchin* (New York, N. Y.).

Schafer, R. D. A generalization of a theorem of Albert. Proc. Amer. Math. Soc. **4**, 452–455 (1953).

The reviewer has shown that every finite power-associative division ring of characteristic $p>5$ is a field. The present note provides a proof of the following generalization. Let R be a finite power-associative ring without elements of additive order 2, 3 or 5 and let every element a of R satisfy an equation of the form $a^{n(a)}=a$ where $n(a)>1$ is an integer. Then either R is a direct sum of finite fields or the attached ring R^+ is a direct sum of components each of which is either a finite field or is the three-dimensional classical

central simple Jordan algebra without nilpotent elements over a finite field. *A. A. Albert* (Chicago, Ill.).

Ward, James A. From generalized Cauchy-Riemann equations to linear algebras. Proc. Amer. Math. Soc. **4**, 456–461 (1953).

Soit A une algèbre associative et commutative de rang n sur le corps des nombres réels, admettant un élément unité e_1, et soit $(e_i)_{1\leq i\leq n}$ une base de A. Un système de n fonctions $y_i(x_1, \cdots, x_n)$ de n variables réelles définit une fonction analytique $\eta=\sum_{i=1}^n y_i e_i$ de la variable $\xi=\sum_{i=1}^n x_i e_i$ si la matrice jacobienne $(\partial y_r/\partial x_s)$ est égale à une matrice de la représentation régulière de A. Pour qu'il en soit ainsi, il faut et il suffit que les y_i satisfassent à un système d'équations aux dérivées partielles $\sum_{i,j} d_{kij}\partial y_i/\partial x_j=0$, où $1\leq k\leq n(n-1)$, les d_{kij} ne dépendant que de la table de multiplication de A et étant telles que $\sum_{i=1}^n d_{kii}=0$ pour tout indice k. L'auteur donne une condition suffisante, trop compliquée pour être reproduite ici, pour qu'inversement un tel système différentiel soit obtenu à partir d'une algèbre commutative A; on peut alors exprimer les solutions d'un tel système au moyen de séries de puissances dans A. *J. Dieudonné.*

★Shih, Kung-sing. Cohomology of associative algebras and spectral sequences. Abstract of a thesis, University of Illinois, Urbana, Ill., 1953. 7 pp.

Let A, B be algebras over a field F with identity elements; let $P=A\otimes_F B$; and let M be a unitary two-sided P-module. Let $C^n=C^n(P, M)$ be the group of normalized n-cochains on P to M, with the usual coboundary formula. Then $C=\sum_0^\infty C^n$ may be filtered by defining C_j^n to be the subgroup of C^n consisting of those n-cochains which vanish when $(n-j+1)$ among the arguments belong to A (embedded in the natural way as a subalgebra of P), and $C_j=\sum_n C_j^n$. The group $C^n(A, M)$ admits left and right operators from B, and so, hence, does $H^n(A, M)$. The author's main theorem asserts that if (E_r) is the spectral sequence derived from the filtration (C_j), then

$$\text{(T)}\qquad \begin{aligned} E_1^{p,q}&\approx C^p(B, H^q(A, M)),\\ E_2^{p,q}&\approx H^p(B, H^q(A, M)). \end{aligned}$$

In §5 of part I of the thesis, the author shows how the argument may be modified to give a direct proof of the theorem on group extensions analogous to (T), due to Hochschild and Serre [Trans. Amer. Math. Soc. **74**, 110–134 (1953); these Rev. **14**, 619]. In §6, the author proves a theorem conjectured by Hochschild and already proved by I. H. Rose [Amer. J. Math. **74**, 531–546 (1952); these Rev. **14**, 130], namely, that if $H^m(A)=0$ for all two-sided A-modules of coefficients and if $H^n(B)=0$ for all two-sided B-modules of coefficients, then $H^{m+n-1}(P)=0$ for all two-sided P-modules of coefficients. However, the conjecture that, in Rose's notation, $\dim P=\dim A+\dim B$ remains open.

Part II of the thesis studies first the more general case where P is an algebra over F with identity, and A is a subalgebra with identity such that P has a free A-basis. A second filtration (C_j^*) is introduced on C which is analogous to that introduced in chapter II of the paper of Hochschild and Serre cited above. By this method, the author is able to show that, if A is separable, then the cohomology groups of P are isomorphic with those of P mod A. The two filtrations on C are correlated by adapting the methods of Hochschild and Serre, and, reverting to the case $P=A\otimes_F B$, the author shows that (T) remains valid with the new filtration. *P. J. Hilton* (Cambridge, England).

Schafer, R. D. The Casimir operation for alternative algebras. Proc. Amer. Math. Soc. **4**, 444–451 (1953).

A representation $[S, T]$ of an alternative algebra A over F is a pair of mappings $x\to S_x$, $x\to T_x$ of A into the algebra of all linear transformations on a vector space such that $S_xT_x-T_xS_x=T_xS_x-S_xT_x=S_{xx}-S_xS_x=T_{xx}-T_xT_x$. The set K_S of all x such that $S_x=0$ is then an ideal of A as is the set K of all x in A such that trace $S_{xa}=0$ for every a of A. Moreover if A is semisimple we have $K=K_S$. Then $A=K_S\oplus H_S$. The function [trace S_{xy}] is a nondegenerate symmetric bilinear form on $H_S\times H_S$, and if $u_1, \cdots, u_m$ is a basis of H_S there is a dual basis $v_1, \cdots, v_m$ such that [trace $S_{u_iv_j}$]$=\delta_{ij}$. The Casimir operation Γ_S is then the endomorphism $\sum_{i=1}^m S_{v_i}S_{u_i}$. It commutes with every S_x and every T_x if F has characteristic zero. In the case where A is associative this operation can be used to give a direct proof of the second Whitehead lemma. The author points out the fact that the lemma is equivalent to the case of the Wedderburn principal theorem for an alternative algebra A with radical N, in which N does not properly contain an ideal $B\neq 0$ and so a direct proof of it is very desirable. He is only able to prove a weaker result which he does use to simplify the known proof of the Wedderburn Principal Theorem for alternative algebras. *A. A. Albert.*

Leger, George F., Jr. A note on the derivations of Lie algebras. Proc. Amer. Math. Soc. **4**, 511–514 (1953).

An algebra A is said to split over a subalgebra B in case there exists a subalgebra C of A such that A is the direct sum $A=B+C$, in which case C is called a complement of B in A. Let L be a Lie algebra of characteristic 0, and R be its radical. Theorem: If the derivation algebra $D(R)$ of R splits over its ideal $I(R)$ of inner derivations of R, then $D(L)$ splits over $I(L)$. An example (due to Hochschild) is given of a nilpotent Lie algebra L which is such that $D(L)$ does not split over $I(L)$. The proof of the theorem depends on the following consequence of a well-known theorem of Malcev and Harish-Chandra [Harish-Chandra, same Proc. **1**, 14–17 (1950); these Rev. **11**, 491]. If T is an ideal of L contained in R, if L splits over T, and if S is any semisimple subalgebra of L, then there is a complement Q of T in L such that Q contains S. *R. D. Schafer* (Storrs, Conn.).

Wolf, Paul. Grundlagen der Theorie der invarianten Kennzeichnung galoisscher Körper mit vorgegebener Galoisgruppe. Math. Nachr. **9**, 201–216 (1953).

The author presents a treatment of the associativity, commutativity and semi-simplicity conditions for Galois algebras K/Ω with the group $\mathfrak{G}=\{S, T, U, \cdots\}$, which avoids the use of group representations and representation modules which were used in the earlier work of Hasse [J. Reine Angew. Math. **187**, 14–43 (1950); these Rev. **11**, 576] and the author [Abh. Math. Sem. Univ. Hamburg **18**, 179–195 (1952); these Rev. **14**, 531]. He introduces directly the invariant of K/Ω in the Kronecker product $G\times G$ of the group ring G of $\mathfrak{G}$ over Ω, which was given by Knobloch [Math. Nachr. **6**, 21–44 (1951); these Rev. **13**, 314]. Basic for this development is a general remark concerning the n-fold product $G\times\cdots\times G=G_n$ on which $\mathfrak{G}$ is to act as follows

$$\begin{aligned}(T_1\times\cdots\times T_{n-1}\times T_n)S&=(T_1\times\cdots\times T_{n-1}\times T_nS),\\ S(T_1\times\cdots\times T_{n-1}\times T_n)&=(ST_1\times\cdots\times ST_{n-1}\times T_n):\end{aligned}$$

All $X\in G_n$ with $XS=SX$ are given as

$$X=(\sum_T(T^{-1}\times\cdots\times T^{-1}\times T))\,(A\times 1)$$

for $A \varepsilon G_{n-1}$. The algebra K/Ω is by definition isomorphic to G as G-right module, where for a fixed isomorphism the unit $1 \varepsilon G$ is mapped on $\Theta \varepsilon K$, so that the elements Θ^R, $R \varepsilon \mathfrak{G}$, form a normal basis of K/Ω. Then the Kronecker product $G_{n-1} \times K$ can be considered as a G_{n-1}-module, and it is as such operator isomorphic to G_n. This observation implies the following generalization of properties of the classical Lagrange resolvent: If $M \varepsilon G_{n-1} \times K$ is termed G_{n-1}-regular provided $MX=0$ with $X \varepsilon G_{n-1}$ implies $X=0$, then all G-regular elements W of $G \times K$ with $W^S = SW$ are obtained from $W = \sum_T T^{-1}\Theta^T$ in the form WA with a regular element $A \varepsilon G$. In order to obtain the new characterization of the Galois algebras K/Ω the author introduces (i) the isomorphism H_i of $G \times K$ into $G_i \times K$ which is the extension of $T \rightarrow T \times \cdots \times T$ (i times) for $T \varepsilon G$, and for which the elements of K are left invariant, and (ii) the Kronecker product $M_1 \times M_2$ for

$$M_1 = (T_1 \times \cdots \times T_{n_1})\Theta^R \varepsilon G_{n_1} \times K$$

and

$$M_2 = (U_1 \times \cdots \times U_{n_2})\Theta^V$$

as

$$(T_1 \times \cdots \times T_{n_1} \times U_1 \times \cdots \times U_{n_2})\Theta^R\Theta^V$$

in $G_m \times K$, $m = n_1 + n_2$. A direct computation which resembles one familiar from the construction of a simple algebra as a crossed product over a maximally commutative subfield shows, using the transformation rules for the cited generalization of the Lagrange resolvent, that $W \times W = W^{H_2}C$ with $C \varepsilon G_2$. This so-called normal factor C is changed into $C^A = A^{-H_2}C(A \times A)$ for regular $A \varepsilon G$ in $W \rightarrow WA$. The author expresses next without an undue amount of computation the associativity, commutativity and semi-simplicity of K/Ω in terms of normal factors. Finally he shows how his factors are related to the explicit factor systems of matrices which were originally introduced by Hasse [loc. cit.]. In the remainder of the paper the concept of irreducibility of normal factors is introduced in a formal manner to characterize algebras which are fields, and the multiplication of abelian algebras is discussed. *O. F. G. Schilling.*

✶**Dürbaum, Hansjürgen. Beiträge zur allgemeinen Bewertungstheorie.** Dissertationen der mathematisch-naturwissenschaftlichen Fakultät der westfälischen Wilhelms-Universität zu Münster in Referaten, Heft 2, pp. 8–9. Aschendorffsche Verlagsbuchhandlung, Münster, 1952.

Theory of Groups

✶**Borůvka, Otakar. Úvod do theorie grup.** [**Introduction to the theory of groups.**] 2d ed. Přírodovědecké Vydavatelství, Prague, 1952. 154 pp. 127 Kčs.

[For a review of the 1st edition see these Rev. 7, 510.] This second edition of Borůvka's "Introduction to the theory of groups" emphasizes still more than the first the tendency to express the fundamental theorems of abstract group theory in the language of set theory by making consistent use of theorems regarding decomposition of sets. It devotes a relatively large space to the discussion of groupoids in which the associative law of the composition of elements is not assumed. The presentation is very clear and the proofs given in all details. The book contains many exercises. *C. Loewner* (Stanford, Calif.).

Šik, F. Sur les décompositions créatrices sur les quasigroupes. Publ. Fac. Sci. Univ. Masaryk **1951**, 169–186 (1951). (Russian summary)

The author develops a theory of the congruence relations, called here creative decompositions, on a (finite) quasigroup G, in terms of the group G_r of permutations on G generated by its multiplications. The main tool is the theorem that a decomposition is creative (an equivalence relation on G is a congruence) if and only if it partitions G into the transitivity systems of an invariant subgroup of G_r. Some attention is paid to the case when the corresponding homomorph of G is a loop. Terminology and concepts due to Borůvka [Math. Ann. **118**, 41–64 (1941); Rozpravy II. Třídy České Akad. **53**, no. 23 (1943); Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **44**, 330–343 (1944); these Rev. **3**, 200; **8**, 449] play a large part in the investigation, which leads to a Jordan-Hölder theorem for quasigroups. Specialisation of G to be a loop yields results contained in those of Albert [Trans. Amer. Math. Soc. **54**, 507–519 (1943); **55**, 401–419 (1944); these Rev. **5**, 229; **6**, 42].

Since it is not stated explicitly that only finite quasigroups are considered, and some remarks could be read as implying that the work is valid for quasigroups generally, it is desirable to point out that the assumption of finiteness is made tacitly (p. 173) in showing that every homomorph of a quasigroup is a quasigroup [cf. Bates and Kiokemeister, Bull. Amer. Math. Soc. **54**, 1180–1185 (1948); these Rev. **10**, 353]. *I. M. H. Etherington* (Edinburgh).

Tamura, Takayuki. Some remarks on semi-groups and all types of semi-groups of order 2, 3. J. Gakugei Tokushima Univ. **3**, 1–11 (1953).

A number of results are proved concerning extensions of semi-groups, and applied to the determination of all semi-groups of orders two and three. *A. H. Clifford.*

Aubert, Karl Egil. On the ideal theory of commutative semi-groups. Math. Scand. **1**, 39–54 (1953).

A subset $\mathfrak{A}$ of a commutative semi-group S is called an s-ideal if $S\mathfrak{A} \subseteq \mathfrak{A}$. The residuated lattice of all s-ideals of S was discussed by Ward and Dilworth [Ann. of Math. (2) **40**, 600–608 (1939); these Rev. **1**, 2]. The present paper amplifies their discussion, and applies it to s-ideals in a commutative ring. *A. H. Clifford* (Baltimore, Md.).

Lorenzen, Paul. Die Erweiterung halbgeordneter Gruppen zu Verbandsgruppen. Math. Z. **58**, 15–24 (1953).

This paper continues earlier work of the author on the theory of divisibility in (non-commutative) semi-ordered groups [Math. Z. **52**, 483–526 (1949); these Rev. **11**, 497] and in semi-ordered sets with operators [ibid. **55**, 269–275 (1952); these Rev. **14**, 127]. In the former paper, it was shown, using transfinite induction, that an r-group G (i.e., a semi-ordered group for which an ideal system H_r has been specified) can be embedded in an r-allowable vector-group Γ if and only if G is r-closed. If Γ_a is the vector-group constructed from all the r-allowable linear orderings of G, then the subset of Γ_a consisting of all finite intersections of elements of G is the ideal-system H_{r_a} of G, which can also be defined intrinsically. The subset V of Γ_a consisting of all finite joins of elements of H_{r_a} is a lattice-ordered group, regular in the sense that $a \wedge xax^{-1} = 1$ implies $a = 1$. It was also shown in the same paper that a regular lattice-ordered group V can be embedded in a vector-group Γ in such a way that V is a sublattice, as well as a subgroup, of Γ. Consequently the first-mentioned theorem could also be

proved by showing directly that G can be embedded in a regular lattice-ordered group V. The present paper carries out this construction without the use of transfinite induction. When applied to the case in which G is the multiplicative group of a field K, semi-ordered by division relative to an integral domain $I\subset K$, and H_r is the system H_d of Dedekind ideals, the construction of V yields the following new result:

$$a_1\wedge\cdots\wedge a_m \underset{d_a}{\dashv} b_1\vee\cdots\vee b_n$$

if and only if

$$1\in\sum_{s=1}^{k}(a_1b_1^{-1},\cdots,a_\mu b_\nu^{-1},\cdots,a_mb_n^{-1})^s.$$

For $n=1$ this reduces to Prüfer's definition of the integral dependence of b_1 on the ideal $(a_1,\cdots,a_m)$.

An analogous construction is carried out for a semi-ordered set B with operators. In this more general case, however, the author begins with an "Überidealbereich" V_ρ of B, i.e., a minimal distributive lattice containing B, and admitting the same operator domain. For, unlike the case for a group, V_ρ is not uniquely determined by the ideal-system of B contained in it. *A. H. Clifford.*

Trevisan, Giorgio. Classificazione dei semplici ordinamenti di un gruppo libero commutativo con n generatori. Rend. Sem. Mat. Univ. Padova **22**, 143–156 (1953).

The author shows how to determine all the ways of making the free commutative group with n generators into an ordered group; this solves Problem 102 of the reviewer's "Lattice theory" [rev. ed., Amer. Math. Soc. Colloq. Publ., v. 25, New York, 1948; these Rev. **10**, 673]. If $n=2$, there is just one non-Archimedean isomorphism-type, and an infinity of Archimedean isomorphism-types, one for each subgroup $m+n\alpha$ of the additive group of real numbers (m, n integers, α irrational). *G. Birkhoff* (Cambridge, Mass.).

Michiura, Tadashi. On partially ordered groups without proper convex subgroups. Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15**, 231–232 (1953).

Let G be a non-trivially partially ordered group, not necessarily commutative. The author shows that if G contains no proper convex subgroup, and if $nx\geqq 0$ for some natural number n and some $x\in G$ implies $x\geqq 0$, then G is isomorphic to a subgroup of the additive group of real numbers. The proof follows a suggestion made by the reviewer to Ky Fan [Ann. of Math. (2) **51**, 409–427 (1950), see p. 426; these Rev. **11**, 525]. *A. H. Clifford.*

Iseki, Kiyoshi, and Michiura, Tadashi. Note on papers by C. J. Everett and L. Fuchs. J. Osaka Inst. Sci. Tech. Part I. **2**, 51 (1950).

The authors wish to withdraw this paper.

Neumann, B. H., and Neumann, Hanna. On a class of abelian groups. Arch. Math. **4**, 79–85 (1953).

H. Yamabe has recently [Proc. Japan Acad. **27**, 205–207 (1951); these Rev. **13**, 818, 1140] studied groups on which an integer-valued bilinear function $\varphi(x,y)$ is defined. The present authors take up the topic. They call a function $\varphi(x,y)$ on an additively written group G a Yamabe function if it satisfies the following conditions:

$$\varphi(x+x',y)=\varphi(x,y)+\varphi(x',y);$$
$$\varphi(x,y+y')=\varphi(x,y)+\varphi(x,y');$$

$\varphi(x,x)=0$ if and only if $x=0$. A group which possesses a Yamabe function is easily seen to be Abelian. One of Yamabe's results was that countable groups with a Yamabe function are free Abelian. The authors prove that the same result holds if the order of the group is $\aleph_1$. They conjecture that groups with a Yamabe function are free Abelian whatever their order. *K. A. Hirsch* (London).

Neumann, B. H. On a problem of Hopf. J. London Math. Soc. **28**, 351–353 (1953).

By considerations involving free products with amalgamations, it is shown that the group $G=\{x,y,z: yx=xy^2, yz=zy\}$ has no elements of finite order. The group

$$H=\{a,b,c: ba=ab^2,\ bc=cb,\ (ab^{-1}a^{-1}c^{-1}aba^{-1}c)^2=1\}$$

has an element of order 2, namely $ab^{-1}a^{-1}c^{-1}aba^{-1}c$ (which is easily shown to be $\neq 1$ in H). Hence G and H are not isomorphic. On the other hand $x\to a$, $y\to b$, $z\to c$ maps G homomorphically on H, and $a\to x$, $b\to y^2$, $c\to z$ maps H homomorphically on G. This surprising example answers in the negative a famous question formulated by H. Hopf: If two finitely generated groups can be mapped homomorphically on each other, are they necessarily isomorphic? The groups in this example are not only finitely generated but even finitely related. *R. H. Fox* (Princeton, N. J.).

✱Szép, Jenő. Über endliche einfache Gruppen. Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 451–453. Akadémiai Kiadó, Budapest, 1952. (Hungarian. Russian and German summaries)

Let G be a group, properly factorized into subgroups H, K, $G=HK$, $H\cap K=1$. If H is simple of composite order, and K abelian, but not elementary, and if the orders of H and K are co-prime, then G is simple if and only if H is a maximal subgroup of G. *K. A. Hirsch* (London).

de Bruijn, N. G. On the factorization of finite abelian groups. Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes. Math. **15**, 258–264 (1953).

The author calls a group G 'good', if a factorisation $G=AB$ into complexes A and B is only possible when at least one factor is periodic, and 'bad' otherwise. A complex C is said to be periodic if, for some element g of G other than the unit element, $gC=C$. He proves the following generalisation of a result of Hajós [Acta Math. Acad. Sci. Hungar. **1**, 189–195 (1950); these Rev. **13**, 623]. Let G have a proper subgroup H which is a direct product $H_1\times H_2$, where the orders of the groups H_1 and H_2 are composite, and neither is isomorphic to the group of order four containing the four elements $1, g_1, g_2, g_1g_2$ ($g_1^2=g_2^2=1$). Then G is bad. He also shows that G is bad if H_1 and H_2 are cyclic and (i) of orders (not necessarily composite) greater than 3, or (ii) each of order 3 when G/H has composite order.

The groups of order 2^m generated by g_i ($i=1,2,\cdots,m$), where $g_i^2=1$, are shown to be good for $m\leqq 5$ and bad for $m>5$. In the case $m=6$ we have the smallest known bad group which is of order 64 and admits a factorisation into two nonperiodic factors each containing eight elements. It is also shown that a subgroup of a good group is good.

By means of these results the author is able to classify a very large number of groups as either good or bad and so extend the work of Hajós and Rédei [see these Rev. **13**, 623]. In particular, for cyclic groups only the following groups are undecided: $\{p^2,q^2\}$, $\{p^2,q,r\}$ and $\{p,q,r,s\}$.

Here p, q, r and s are different primes and the first group, for example, is the direct product of two cyclic groups of orders p^2 and q^2. *R. A. Rankin* (Birmingham).

Kulikov, L. Ya. Generalized primary groups. II. Trudy Moskov. Mat. Obšč. 2, 85–167 (1953). (Russian)

[Part I appeared in same Trudy 1, 247–326 (1952); these Rev. 14, 132.] Let G be a generalized primary group (that is, a module over K_p or Z_p in the notation of the cited review). We define a transfinite series of subgroups G_α as follows: for α a limit ordinal, G_α is the intersection of the preceding subgroups, and otherwise $G_\alpha = \bigcap p^n G_{\alpha-1}$. These subgroups settle down to the maximal divisible subgroup of G, and if the latter is 0 we say that G is reduced. The groups $A_\alpha = G_\alpha / G_{\alpha-1}$ are called the Ulm invariants of G; they are groups with no elements of infinite height. The bulk of the paper (§§5–10) is devoted to the proof of Theorem 10.1, which gives necessary and sufficient conditions for the existence of a reduced group with prescribed cardinal number m and Ulm invariants A_α. These conditions are as follows:

$$(1)\qquad \sum_\alpha |A_\alpha| \leq m \leq \prod_n |A_n|,$$

$$(2)\qquad |(p^n A_\alpha)[p]| \geq |A_{\alpha+1} - pA_{\alpha+1}|,$$

$$(3)\qquad |A_\alpha|^{\aleph_0} \geq \sum_{\alpha \leq \nu} A_\nu.$$

Here the vertical bars denote cardinal number, α and ν run over the ordinals occurring in the given series A_α, n runs over the integers, and the notation $G[p]$ calls for taking the subgroup of G consisting of all elements x satisfying $px=0$. Using this result the author is able to sharpen a theorem of Kurosch [Mat. Sbornik N.S. 5(47), 347–354 (1939); these Rev. 2, 2] asserting that there are precisely 2^m non-isomorphic groups of cardinal number m; he proves that there are already that many abelian groups. By duality there is a corresponding enumeration of compact abelian groups.

The final section gives a generalization of Ulm's theorem. One assumes that G is a reduced generalized primary group which satisfies three hypotheses: (1) G is countably generated (as a module over K_p or Z_p), (2) each Ulm invariant is a direct sum of cyclic groups, (3) if E is a finitely generated subgroup of G, then there are only a finite number of indices α such that $E \cap G_\alpha \neq E \cap G_{\alpha+1}$. It is immediate that ordinary primary groups satisfy these hypotheses, so that Ulm's original theorem is subsumed. Known examples show that hypotheses (1) and (3) cannot be dropped, but the author leaves open the possibility of omitting (2).

I. Kaplansky (Chicago, Ill.).

Schenkman, Eugene. A generalization of the central elements of a group. Pacific J. Math. 3, 501–504 (1953).

The author calls an element a of a group G weakly central if for every g in G, $(a, a, \cdots, a, g) = 1$ provided the commutator is of sufficiently high degree, or equivalently, if $a^{(n)}(g) = a$ for some n, where $a^{(n)}(g)$ is defined recursively by the formulas $a^{(1)}(g) = g^{-1}ag$, $a^{(n)}(g) = a(a^{(n-1)}(g))$. He then proves the following two theorems by elementary methods for finite G, and extends them to locally finite G. (1) If for a fixed prime p all the elements of order p in G are weakly central, then they constitute a normal subgroup of G. (2) If G is solvable, then the weakly central elements constitute a normal subgroup of G which is a direct product of primary groups. Theorem (1) implies immediately the following group theoretical analogue to Engel's theorem for Lie algebras: If all the elements of G are weakly central, then G is the direct product of primary groups (and hence is nilpotent in the finite case). It is not known whether solvability is necessary for theorem (2). *D. G. Higman.*

Gonçalves, J. Vicente. On groups having a set of p-Sylow subgroups. Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. (2) 2, 161–168 (1952).

This paper contains the following generalization of Burnside's theorem on the normalizer of the maximal intersection of p-Sylow subgroups, as well as some remarks on p-invariants. Let $H = H_1, H_2, \cdots, H_\alpha$ be all the p-Sylow subgroups of a finite group G. If $D = H \cap H_2$ is not properly contained in any $H \cap H_i$, then the intersection with H of the normalizer N_D of D is a p-Sylow subgroup of N_D, and the number of p-Sylow subgroups of N_D is greater than one, and equal to the index of $N_D \cap H$ in the normalizer of H.

D. G. Higman (Montreal, Que.).

Raševskiĭ, P. K. On some fundamental theorems of the theory of Lie groups. Uspehi Matem. Nauk (N.S.) 8, no. 1(53), 3–20 (1953). (Russian)

The author's purpose is to give proofs for several theorems of Lie theory, which have not been available in the Russian literature, namely, for the theorem of Levi, with Malcev's refinement, and for the theorem on complete reducibility of representations of semi-simple groups. These theorems are deduced from the following new theorem: An affine (linear, non-homogeneous) representation of a semi-simple Lie algebra always has a fix-point. Two proofs for this are given. (I) For a compact (i.e., definite Cartan form) real Lie algebra one utilizes the corresponding simply connected compact Lie group; the induced representation obviously has a fix-point. The case of a complex representation of a complex Lie algebra is easily reduced to the consideration of the compact form. For a real non-compact Lie algebra one complexifies the Lie algebra and the representation. (II) Algebraic proof: One can assume, by induction, that the induced homogeneous-affine representation is irreducible. The author then proves the known fact that the Casimir operator of this homogeneous representation is a non-zero (positive rational) multiple of the identity. The fix-point can then be written down explicitly. Complete reducibility now follows easily: The subspaces complementary to an invariant subspace form a linear space, in which a non-homogeneous affine transformation is induced; the fix-point represents an invariant subspace. The proofs of the Levi-Malcev theorems are not quite so immediate; they require division into cases according to the behavior of the radical, and induction over the dimension.

H. Samelson (Princeton, N. J.).

Abdelhay, José. Representation of locally compact groups. Revista Cientifica 3, no. 3–4, 3–55 (1952). (Portuguese)

Expository paper based on lectures by W. Ambrose. Contents: irreducible unitary representations of a locally compact group G and their relations with positive definite functions on G (Gelfand-Raikov), the existence of sufficiently many such representations being proved by the Krein-Milman theorem. The theory is then specialized to abelian groups, pushed as far as the definition of the dual group $\hat{G}$, and of the Fourier transform of a function of $L^1(G)$, and concludes with the proof of Bochner's theorem.

J. Dieudonné (Evanston, Ill.).

On order

***Dieudonné, J.* Análise harmônica. [Harmonic analysis.]** Notes prepared by J. Abdelhay. Universidade do Brasil, Rio de Janeiro, 1952. iii+212 pp.

This book consists of notes compiled from lectures by Dieudonné at the University of Brazil and translated into Portuguese and edited by J. Abdelhay. Chapter I contains a complete discussion of duality and Fourier transforms for finite Abelian groups, and sketchy descriptions of these theories for the additive group of integers, the circle group, and the additive real numbers. Chapter II is devoted to a miscellany of preliminary material: the elementary theory of Banach algebras, including the Gel'fand-Naĭmark theorem; measures and integrals associated with locally compact Hausdorff spaces; Hilbert spaces, including the spectral theorem for bounded Hermitian operators. Chapter III is devoted primarily to a study of commutative Banach algebras A with unit and involution $x\rightarrow\bar{x}$ for which a positive definite linear functional f exists ($f(\bar{x})=\overline{f(x)}$ and $f(x\bar{x})\geqq 0$ for all $x\in A$). Plancherel's theorem is presented in a form due to Godement asserting that under certain conditions A is isomorphic to an algebra of continuous complex functions $\hat{x}$ vanishing at infinity on a locally compact Hausdorff space Σ, and that $f(xy)=\int_\Sigma \hat{x}\hat{y}d\mu$ for all $x, y\in A$ and some positive measure μ on Σ. A number of special properties of positive definite functions are also analyzed. Chapter IV deals with harmonic analysis on a locally compact Abelian group G. The bounded Borel measures $\mathfrak{M}_1(G)$ on G under convolution and ordinary addition are studied as the group algebra of G_1 and $L_1(G)$ is treated as a closed ideal in $\mathfrak{M}_1(G)$. The relevant computations are given in great detail. The Plancherel transform is defined, by using the Plancherel-Godement theorem, for all positive definite measures on G, and the usual formulas are easy consequences. Strangely, nowhere does one find an explicit statement that $L_2(G)$ and $L_2(\hat{G})$ are unitarily equivalent under the Fourier transform.

This book is not a complete treatise on harmonic analysis, nor is it an introduction suitable for beginners. Such important matters as the existence of Haar's measure are disposed of by references to standard treatises. There are no fewer than 50 references to the works of Bourbaki, Banach, Weil, and Pontryagin necessary to follow certain proofs. Comparisons are inevitable with the recent book of L. H. Loomis [An introduction to harmonic analysis, Van Nostrand, New York, 1953; these Rev. **14**, 883]. Loomis's volume is far ahead on completeness and general readability, but the book under review has compensations for the expert in the novelty of approach (for which the author credits H. Cartan and Godement) and in the many interesting details included. *E. Hewitt* (Seattle, Wash.).

NUMBER THEORY

Gloden, A. Une méthode de résolution de l'équation diophantienne $2x^2+1=ay^2$, $(a=2b^2+1)$. Bull. Soc. Roy. Sci. Liège **22**, 195–196 (1953).

Ostrom, T. G. Concerning difference sets. Canadian J. Math. **5**, 421–424 (1953).

The author derives several further consequences of M. Hall's Multiplier Theorem utilizing results of Hall [Duke Math. J. **14**, 1079–1090 (1947); these Rev. **9**, 370], Evans and Mann [Sankhyā **11**, 357–364 (1951); these Rev. **13**, 899] and Mann [Canadian J. Math. **4**, 222–226 (1952); these Rev. **14**, 5]. For instance, the following interesting result: If $n=m^r$, $(r, 3)=1$, and there is a difference set mod $N=n^2+n+1$, then there is a difference set

$$\text{mod } N_1=m^2+m+1$$

and every multiplier mod N is a multiplier mod N_1.
H. B. Mann (Columbus, Ohio).

Lehmer, Emma. On residue difference sets. Canadian J. Math. **5**, 425–432 (1953).

If $p=Kn+1$ is a prime then there are K nth power residues mod p. The author is concerned with the case that these nth power residues form a difference set (D.S.) mod p. If this is the case then the multiplicity of this D.S. is $\lambda=(K-1)/n$. The author proves the following results: A necessary and sufficient condition for this case to arise is that the cyclotomic numbers $(i, 0)$ are all equal. There is no residue difference set if n is odd or if n and k are both even. Residue D.S.'s with $\lambda=1$, $p\leqq 2{,}561{,}600$ arise only for $p=1$ and $p=73$. The biquadratic residues form a D.S. if and only if $p=1+4y^2$, $y\equiv 1(2)$. [S. Chowla, Proc. Nat. Acad. Sci. India. Sect. A. **14**, 45–46 (1944); these Rev. **7**, 243]. The octic residues form a D.S. if and only if $(p-1)/8$ and $(p-9)/64$ are odd squares. The 6th residues never form a D.S. The 10th residues do not form a D.S. if 2 is quintic residue. The author also considers D.S.'s formed by nth powers and 0. Necessary and sufficient conditions for such D.S.'s are found for $n=4$ and $n=8$. *H. B. Mann.*

Carlitz, L., and Riordan, J. Congruences for Eulerian numbers. Duke Math. J. **20**, 339–343 (1953).

Following Shanks [Amer. Math. Monthly **58**, 404–407 (1951); these Rev. **13**, 899] the authors put

$$\binom{x}{i}^k=\sum_{s=1}^{(k-i+1)} A_{ks}^{(i)}\binom{x+s-1}{ik} \quad (k>0),$$

where for $i=1$ the numbers $A_{ks}=A_{ks}^{(1)}$ are Eulerian numbers. The object of this paper is to prove certain periodic properties of the residues of the $A_{ks}^{(i)}$. These depend upon some formulas due to Frobenius [S.-B. Preuss. Akad. Wiss. **1910**, 809–847]. The main theorem is as follows: For fixed s in the range $p^{j-1}<s\leqq p^j$, $A_{ks}^{(i)}$ with $k\geqq e$ has the period $b=p^{j+e-1}(p-1)$ (mod p^e). *A. L. Whiteman.*

Carlitz, L. Some theorems on Kummer's congruences. Duke Math. J. **20**, 423–431 (1953).

Let p be a fixed prime and let $\{a_m\}$ be a sequence of rational numbers that are integral (mod p). If

$$(*) \qquad \sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}a_{m+s(p-1)}a_p^{r-s}\equiv 0 \pmod{p^r}$$

holds for all $m\geqq r\geqq 1$, then (*) is called Kummer's congruence for $\{a_m\}$. The author first proves the following two theorems. (1) If $\{a_m\}$ satisfies (*) and $\{b_m\}$ satisfies a like congruence, then the same is true for $\{c_m\}=\{a_mb_m\}$. (2) Let $c_m^{(k)}=m^ka_m$, $k\geqq 1$. If $\{a_m\}$ satisfies (*), then (*) also holds with $a_{m+s(p-1)}$ replaced by $c_{m+sp(p-1)}^{(k)}$. Extensions of these theorems in various directions are then obtained. Applications to Euler numbers, Bernoulli numbers, Stirling numbers of the second kind and to the coefficients of the Jacobi elliptic functions are also given. Finally the author considers Kummer's congruence for the sequence $\{c_m\}$, where

$c_m=\sum_{s=0}^{m}\binom{m}{s}a_s b_{m-s}$. Put

$$f(x)=\sum_1^\infty a_m x^m/m!,\quad g(x)=\sum_1^\infty b_m x^m/m!.$$

Assuming that $a_p\equiv b_p \pmod p$, and

$$(D^p-a_pD)f(x)=p\sum_0^\infty A_m f^m(x),$$

where the A_m are integral (mod p), and a like formula for $g(x)$, the author proves that congruence (*) holds with $a_{m+s(p-1)}$ replaced by $c_{m+s(p-1)}$. Under the milder assumption $f'(x)=\sum_0^\infty A_m f^m(x)$ and a like assumption for $g(x)$, the modulus p^r must be replaced by p^{r_1}, where r_1 is the greatest integer $\leqq\frac{1}{2}(r+1)$. *A. L. Whiteman* (Princeton, N. J.).

***Obláth, Richard. Quelques propriétés arithmétiques des radicaux.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 445–450. Akadémiai Kiadó, Budapest, 1952. (Hungarian. Russian and French summaries)

The author proves among others the following theorem: Let a, b, c, m, n, r be rational integers. Then $a^{1/m}+b^{1/n}=c^{1/r}$ is impossible except if $a^{n/m}=\beta$ and $a^{r/m}=\gamma$ are rational integers and $b=b_1{}^n\beta$, $c=(1+b_1)^r\gamma$ where b_1 is a rational number. *P. Erdös* (South Bend, Ind.).

Ansari, A. R. On prime representing function. Ganita **2**, 81–82 (1951).

It was established by W. H. Mills [Bull. Amer. Math. Soc. **53**, 604 (1947); these Rev. **8**, 567] that there exists a real number A such that $[A^{3^x}]$ is a prime for every positive integer x. The present paper replaces the "3" by any $c>77/29$. This improves the reviewer's $c>8/3$ [Proc. Amer. Math. Soc. **2**, 753–755 (1951); these Rev. **13**, 321]. It is also shown that this can be extended to $c>2$ under the Riemann hypothesis. *I. Niven* (Eugene, Ore.).

Sierpiński, W. Sur une propriété des nombres premiers. Bull. Soc. Roy. Sci. Liège **21**, 537–539 (1952).

Let p_n be the nth prime with $p_1=2$. Two "empirical theorems" were given by H. F. Scherk [J. Reine Angew. Math. **10**, 201–208 (1833); cf. Dickson, History of the theory of numbers, v. 1, Carnegie Inst. of Washington, 1919, p. 436] as follows. It is possible to choose signs such that

$$p_{2n}=1\pm p_1\pm\cdots\pm p_{2n-2}+p_{2n-1},$$
$$p_{2n+1}=1\pm p_1\pm\cdots\pm p_{2n-1}+2p_{2n}.$$

The author shows that these conjectures can be deduced in a simple way from Chebyshev's theorem (Bertrand's postulate) that $p_{n+1}<2p_n$. The fact that Scherk's results have not been proved before is perhaps due to the fact that they are rather special and that Scherk failed to notice or mention that the left sides of his equations need not be primes.

The author proves by induction on $n\geqq 3$ that every odd number $\leqq p_{2n+1}$ is of the form $\pm p_1\pm p_2\pm\cdots\pm p_{2n-1}+p_{2n}$. In fact, this follows when the primes $>17=p_7$ are replaced by any increasing sequence of odd numbers for which each is less than twice its predecessor. From this result Scherk's observations are easily deduced. *D. H. Lehmer.*

Chapron, R. Mémoire sur une famille de conditions nouvelles de primalité. J. Chinese Math. Soc. (N.S.) **1**, 343–376 (1951). (Chinese summary)

The author combines Wilson's theorem with Legendre's theorem on the highest power of 2 dividing $(p-1)!$, to obtain the following result. Let J denote the largest odd factor of $(p-1)!$ and s the sum of the digits of p when written in the binary system. Then a necessary and sufficient condition for the primality of p is that p divide $J+2^s$. He claims that this condition is simpler than Wilson's criterion: p is a prime if and only if p divides $1+(p-1)!$ because J is "very much smaller than" $(p-1)!$. The author makes the mistake made by nearly all commentators on Wilson's criterion. It is not the size of $(p-1)!$ which makes Wilson's theorem impractical, but rather the fact that the factorial function has no practical duplication formula.

The author generalizes his results by replacing 2 by any integer less than p. Examples are discussed for extremely small primes. *D. H. Lehmer* (Los Angeles, Calif.).

Kober, H. A remark on zeta functions. Proc. Amer. Math. Soc. **4**, 588–590 (1953).

In section 1, the author obtains the following expression for Riemann's ξ-function:

$$(1)\quad \xi(s)=F_s(w)+F_{1-s}(1/w)\quad (w=u+iv,\ u\geqq 0,\ w\neq 0)$$

where $F_s(w)=\int_0^\infty\omega(x+w)(x+w)^{\frac{1}{2}s-1}dx-w^{\frac{1}{2}(s-1)}/(1-s)$, and $\omega(x)=\sum_{n=1}^\infty\exp(-n^2\pi x)$, thus generalizing the classic formula $\xi(s)=\int_0^\infty\omega(x)x^{\frac{1}{2}s-1}dx$. Formula (1) is then used to obtain two necessary and sufficient conditions for the non-trivial vanishing of the Riemann zeta function.

In section 2, the author considers the generalized zeta function $\zeta(s,a,b)$ defined by analytic continuation of the series $\sum_{n>-a}e^{2\pi inb}(n+a)^{-s}$ (a, b arbitrary real numbers), and sketches a proof of the functional equation

$$\frac{(2\pi)^s}{\Gamma(s)}e^{2\pi iab}\zeta(1-s,a,b)=e^{\frac{1}{2}\pi is}\zeta(s,b,-a)+e^{-\frac{1}{2}\pi is}\zeta(s,-b,a).$$

For $0<a\leqq 1$, $0<b<1$, the result is equivalent to a transformation formula of Lerch [see Acta Math. **11**, 19–24 (1887)]. *T. M. Apostol* (Pasadena, Calif.).

Val'fiš, A. Z. On Euler's function. Doklady Akad. Nauk SSSR (N.S.) **90**, 491–493 (1953). (Russian)

Let $R(x)=\sum_{n\leqq x}\phi(n)-3x^2/\pi^2$ where $\phi(n)$ is the Euler ϕ-function. It is well-known that $R(x)=O(x\log x)$, a result due to Mertens. The author, using estimation techniques of the Vinogradoff variety, establishes the better estimate $R(x)=O(x(\log x)^{3/4}(\log\log x)^2)$. *R. Bellman.*

Pérez-Cacho, L. On the function $E(x)$ (integral part of x). Revista Mat. Hisp.-Amer. (4) **13**, 188–195 (1953). (Spanish)

The author shows how a number of known results concerning sums of the form $\sum_{x=1}^n f(x)E(n/x)$ may be derived in a systematic fashion, and deduces corresponding results for the function $q(n)$ which is the number of odd terms in the sequence $\{E(n/x)\}$, $x=1, 2, \cdots, n$. *R. Bellman.*

Richert, Hans-Egon. Verschärfung der Abschätzung beim Dirichletschen Teilerproblem. Math. Z. **58**, 204–218 (1953).

The author applies a two-dimensional version of the Van der Corput method to the estimation of the remainder term, $\Delta(x)$, in the Dirichlet divisor problem. The method was previously applied to the circle problem by Titchmarsh [Proc. London Math. Soc. (2) **38**, 96–115 (1934)] and Hua [Quart. J. Math., Oxford Ser. **13**, 18–29 (1942); these Rev. **4**, 190]. The problem is essentially more difficult in the present case because of the possible vanishing of a crucial

determinant, a Hessian, and the appearance of hyperbolic rather than circular sectors. Using a method followed by Min [Trans. Amer. Math. Soc. **65**, 448–472 (1949); these Rev. **11**, 84] to overcome the first difficulty and some non-trivial estimates to meet the second, the author obtains the estimate $\Delta(x)=O(x^{15/46}\log^{30/23}x)$ which is superior to the best previous, $\Delta(x)=O(x^{27/82})$, derived recently by Nieland.

R. Bellman (Santa Monica, Calif.).

Carlitz, L. Note on some partition identities. Proc. Amer. Math. Soc. **4**, 530–534 (1953).

Let the numbers $p_k(m)$ be defined by

$$\prod_{n=1}^{\infty}(1-x^n)^k=\sum_{m=0}^{\infty}p_k(m)x^m.$$

The author proves: if r is a prime >3 and either $r\equiv3 \pmod 4)$ or $r\equiv5 \pmod{12}$ then

$$\sum_{m=0}^{\infty}p_2(rm+r_0)x^m=\epsilon\prod_{n=1}^{\infty}(1-x^{rn})^2,$$

where $r_0=(r^2-1)/12$ and $\epsilon=+1$ in the first case, $\epsilon=-1$ in the second case. Similar identities are proved for $p_6(m)$ if $r\equiv3 \pmod 4)$, $r\geqq3$ and for $p_4(m)$ if $r\equiv5 \pmod 6)$. Special cases of these identities had previously been given by M. Newman [Trans. Amer. Math. Soc. **73**, 313–320 (1952); these Rev. **14**, 250]. The author also derives another formula given by Newman, viz.

$$\sum_{m=0}^{\infty}p_6(5m)x^m=\prod_{n=1}^{\infty}(1-x^n)^6(1-x^{5n})^{-1}$$

from a result of Ramanujan. *H. D. Kloosterman.*

Carlitz, L. Some theorems on generalized Dedekind sums. Pacific J. Math. **3**, 513–522 (1953).

For a fixed odd integer $p>1$, define

$$G_p(x)=\sum_{m,n=1}^{\infty}n^{-p}x^{mn}=\sum_{n=1}^{\infty}n^{-p}\frac{x^n}{1-x^n}\quad(|x|<1),$$

and let $\tau'=(h'\tau+k')/(k\tau-h)$, $hh'+kk'+1=0$, be a unimodular substitution. Then G_p satisfies the transformation formula [see Apostol, Duke Math. J. **17**, 147–157 (1950); these Rev. **11**, 641]

$$G_p(e^{2\pi i\tau})=(k\tau-h)^{p-1}G_p(e^{2\pi i\tau'})+\frac{(2\pi i)^p}{2(p+1)!}f(h,k;\tau).$$

The function $f(h,k;\tau)$ is a polynomial in τ given by

$$f(h,k;\tau)=\sum_{r=0}^{p+1}\binom{p+1}{r}(k\tau-h)^{p-r}c_r(h,k),$$

the coefficients $c_r(h,k)$ being expressible in the form

$$c_r(h,k)=\sum_{\mu=1}^{k}P_{p+1-r}(\mu/k)P_r(h\mu/k),$$

where $P_r(x)=B_r(x-[x])$ is the rth Bernoulli function. Using this transformation formula, the author derives a reciprocity law for the sums $c_r(h,k)$ which can be expressed symbolically as follows:

$$\binom{p+1}{r}k^r(c(h,k)-h)^{p+1-r}$$
$$=\binom{p+1}{r+1}h^{p-r}(c(k,h)-k)^{r+1}+kB_{p+1-r}B_r-hB_{p-r}B_{r+1},$$

where $c^n(h,k)$ must be replaced by $c_n(h,k)$ after expanding, and the B's are Bernoulli numbers. When $r=p$ the sums $c_r(h,k)$ reduce to the generalized Dedekind sums $s_p(h,k)$ considered previously by the reviewer, the case $p=1$ being due to Dedekind.

Further properties of the sums $c_r(h,k)$ are derived, of which we mention the following formula which gives a representation of these sums in terms of the Eulerian numbers:

$$(1)\quad c_r(h,k)=\frac{B_{p+1-r}B_r}{k^p}+\frac{r(p+1-r)}{k^p}\sum_{t=1}^{k-1}\frac{H_{p-r}(\zeta^{ht})H_{r-1}(\zeta^{-t})}{(\zeta^{-ht}-1)(\zeta^t-1)},$$

where $\zeta=e^{2\pi i/k}$ and $H_m(\rho)$ is defined for $\rho^k=1$, $\rho\neq1$, by $(1-\rho)/(e^t-\rho)=\sum_{m=0}^{\infty}H_m(\rho)t^m/m!$. *T. M. Apostol.*

Carlitz, L. The reciprocity theorem for Dedekind sums. Pacific J. Math. **3**, 523–527 (1953).

Using the Lagrange interpolation formula, the author obtains the identity

$$(1)\quad \frac{1}{k}\sum_{\zeta^k=1}\frac{\zeta}{x-\zeta}\frac{\zeta-1}{\zeta^h-1}+\frac{1}{h}\sum_{\eta^h=1}\frac{\eta}{x-\eta}\frac{\eta-1}{\eta^k-1}$$
$$=\frac{x-1}{(x^h-1)(x^k-1)}-\frac{1}{hk(x-1)},$$

where ζ and η run through the kth roots of unity distinct from 1. This identity is used to obtain a simple proof of the reciprocity law for the Dedekind sums

$$s(h,k)=\sum_{r\bmod k}((r/k))((hr/k)),\quad ((x))=x-[x]-\tfrac12.$$

By expanding $((hr/k))$ into what amounts to a finite Fourier series the sums $s(h,k)$ are expressed as follows:

$$(2)\quad s(h,k)=\frac{1}{4k}+\frac{1}{k}\sum_{\zeta^k=1}\frac{\zeta}{\zeta-1}\frac{1}{\zeta^h-1}.$$

Putting $x=1+t$ in (1), expanding both members in powers of t and comparing coefficients leads at once to the reciprocity law in question:

$$12hk\{s(h,k)+s(k,h)\}=h^2+3hk+k^2+1.$$

By a more recondite application of (1), the author also obtains the reciprocity law for the generalized Dedekind sums $s_p(h,k)$ mentioned in the previous review, using formula (1) of that review in place of (2).

T. M. Apostol (Pasadena, Calif.).

Mardžanišvili, K. K. On an asymptotic formula of the additive theory of prime numbers. Soobščeniya Akad. Nauk Gruzin. SSR. **8**, 597–604 (1947). (Russian)

The author refines a previous paper of his [Izvestiya Akad. Nauk SSSR. Ser. Mat. **4**, 193–214 (1940); these Rev. **2**, 250] concerned with the number $I=I(N_1,\cdots,N_n;s)$ of solutions $(p_1,\cdots,p_s)$ in primes of the system of equations

$$(*)\quad N_k=p_1^k+\cdots+p_s^k,\quad k=1,\cdots,n$$

for given $s, N_1,\cdots,N_n$. His final result is that the asymptotic formula, for large n, which he obtained previously for $s>s_0\sim5n^2\log n$ also holds if $s>s_1\sim(25/2)n^2\log n$.

This improvement by a factor of about n has been made possible by the work of Vinogradov and others on exponential sums. Thus, with the aid of such work, the author proves the following lemma which is basic for the improved value of s. The number of integral solutions $(x_1,\cdots,x_{2l})$ of the system of equations

$$x_1^k+\cdots+x_l^k=x_{l+1}^k+\cdots+x_{2l}^k,\quad k=1,\cdots,n,$$

subject to the conditions $0 \leqq x_j \leqq P$ is $O(P^{2l-n(n+1)/2})$ provided $n>11$ and $l>(s_1-3)/2$; this result is almost the same as a later result of Vinogradov [ibid. 15, 109–130 (1951), Lemma 7; these Rev. 13, 328]. In the proof of this result, Hua's estimate

$$\sum_{x=1}^{q} e^{2\pi i\{f(x)/q\}} = O(q^{1-(1/k)+\epsilon}),$$

$f(x)$ a polynomial of degree k, is used.

Since the appearance of this paper the author has published two papers in which extensions and improvements are stated [Doklady Akad. Nauk SSSR (N.S.) 70, 381–383 (1950); Uspehi Matem. Nauk (N.S.) 4, no. 1(29), 183–185 (1949); these Rev. 11, 502, 872, 161]. See also the paper reviewed below. *L. Schoenfeld* (Urbana, Ill.).

Mardžanišvili, K. K. On a generalization of Waring's problem. Soobščeniya Akad. Nauk Gruzin. SSR 11, 82–84 (1950). (Russian)

In this paper are stated the analogs of the theorems stated by the author in a previous paper [Doklady Akad. Nauk SSSR (N.S.) 70, 381–383 (1950); these Rev. 11, 502, 872]. Whereas the former paper restricted the variables (in (*) of the preceding review) to prime numbers, the present paper allows the variables to be any positive numbers. *L. Schoenfeld* (Urbana, Ill.).

Mardžanišvili, K. K. On some additive problems of the theory of numbers. Acta Math. Acad. Sci. Hungar. 2, 223–227 (1951). (Russian)

This is a brief expository paper giving the history of the Waring problem, the Goldbach problem and such extensions as the Waring-Goldbach problem and the simultaneous system analogs. No references to the literature are given. The exposition very closely follows that in the author's paper appearing in Uspehi Matem. Nauk (N.S.) 5, no. 1(35), 236–240 (1950) [these Rev. 11, 502]. *L. Schoenfeld* (Urbana, Ill.).

***Földes, István. On the Goldbach hypothesis concerning the prime numbers of an arithmetical progression.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 473–492. Akadémiai Kiadó, Budapest, 1952. (Hungarian. Russian and English summaries)

Using the methods of Vinogradov and Hardy-Littlewood the author proves the following theorems. 1) Let $k \geqq 1$, $(g_j, k)=1$ $(j=1, 2, 3)$. Denote by $N(n)$ the number of representations of $n=p_1+p_2+p_3$, p_j prime, $p_j \equiv g_j \pmod{k}$, $j=1, 2, 3$. Then if $n \equiv g_1+g_2+g_3 \pmod{k}$, n odd,

$$N(n)=\frac{k}{\phi(k)^2}\prod_{p|k,\, p\nmid n}\left(1-\frac{1}{p^2-3p+3}\right) \times\prod_{p\nmid k}\left(1+\frac{1}{(p-1)^3}\right)\frac{n^2}{2(\log n)^3}+o\left(\frac{n^2}{(\log n)^3}\right).$$

2) Let $k_j \geqq 1$, $(k_j, k_{j'})=1$, $(g_{j'}, k_j)=1$ $(j, j'=1, 2, 3)$. Let $N(n)$ denote the number of solutions of $n=p_1+p_2+p_3$, $p_j \equiv g_j \pmod{k_j}$. Then

$$N(n)=\phi(n)\frac{n^2}{2(\log n)^3}+o\left(\frac{n^2}{(\log n)^3}\right)$$

where $\phi(n)$ is given by a complicated expression.

These results sharpen previous results of Van der Corput [Math. Ann. 116, 1–50 (1938)]. *P. Erdös.*

Rieger, G. J. Über eine Verallgemeinerung des Waringschen Problems. Math. Z. 58, 281–283 (1953).

The purpose of this note is to give an elementary proof of the following result of Schnirelmann [Math. Ann. 107, 649–690 (1933)]: Let $\mathfrak{M}$ be a monotone sequence of natural numbers including zero and having positive density $d(\mathfrak{M})$. Then the sequence of nth powers of numbers of $\mathfrak{M}$ forms a basis for the natural numbers. A purely arithmetic proof for the case $d(\mathfrak{M})=1$ was given by Linnik [Mat. Sbornik N.S. 12(54), 225–230 (1943); these Rev. 5, 200; see also Khinchin, Three pearls of number theory, Greylock Press, Rochester, N. Y., 1952; these Rev. 13, 724]. A modification of his proof yields the Schnirelmann theorem. *R. D. James.*

Watson, G. L. A simple proof that all large integers are sums of at most eight cubes. Math. Gaz. 37, 209–211 (1953).

Carlitz, L. A theorem of Stickelberger. Math. Scand. 1, 82–84 (1953).

Skolem [Norsk Mat. Tidsskr. 34, 81–85 (1952); these Rev. 14, 251] proved the following Theorem A. Let $f(x)$ be a polynomial of degree n with rational integral coefficients and discriminant D. If p is an odd prime, $p \nmid D$, and $f(x) \equiv f_1(x) \cdots f_s(x) \pmod{p}$, where the $f_i(x)$ are irreducible $\pmod{p}$, then $(D/p)=(-1)^{n-s}$, where (D/p) is the Legendre symbol. This theorem is contained in a theorem of Stickelberger [Verh. 1. Internat. Math.-Kongress., Zürich, 1897, Teubner, Leipzig, 1898, pp. 182–193]. The present author sketches a simple proof of the following supplementary Theorem B. In the notation of Theorem A with $p=2$, we have $(D/2)=(-1)^{n-s}$, where $(D/2)$ denotes the Kronecker symbol. *A. L. Whiteman* (Princeton, N. J.).

Carlitz, L. Distribution of primitive roots in a finite field. Quart. J. Math., Oxford Ser. (2) 4, 4–10 (1953).

In this paper various results about the number of numbers of a certain form and belonging to a given exponent e in the finite field $GF(p^n)$ are given. These results generalize Davenport's theorem that $\Theta+a$ is a primitive root if Θ is a generating element of the field and a a suitable integer provided that p is sufficiently large. For the proofs the author uses a lemma of Davenport about character sums, a method which is a modification of Vinogradov's method for finding the smallest primitive root, and furthermore a theorem of A. Weil. The following results may be mentioned. The number of numbers in $GF(p^n)$ of the form $\Theta+a$ belonging to the exponent e is

$$\frac{\phi(e)}{p^n-1}p+O[p^{1-\{1/2(n+1)\}+\epsilon}] \quad (p\rightarrow\infty).$$

The number of numbers in $GF(p^n)$ of the form

$$\Theta^r+a_1\Theta^{r-1}+\cdots+a_r \quad (a_i \in G(p))$$

is

$$N_r(e)=\frac{\phi(e)}{p^n-1}p^r+O[p^{\frac{1}{2}r+\epsilon}] \quad (p\rightarrow\infty)$$

and is for all p

$$N_r(e)=\frac{\phi(e)}{p^n-1}p^r+O(p^{\frac{1}{2}n+\epsilon}) \quad (p^n\rightarrow\infty).$$

The number of integers t $(s<t\leqq s+r,\, s\geqq 0,\, r\geqq 1)$ which belong to the exponent e is

$$\frac{\phi(e)}{p-1}r+O[p^{\frac{1}{2}}\log p\sum_{d|e}\mu^2(d)].$$

H. Bergström (Göteborg).

Rosenthall, E. Diophantine equations separable in cyclotomic fields. Duke Math. J. **20**, 217–232 (1953).

The author generalizes his method for solving certain cubic diophantine equations by means of corresponding multiplicative ideal equations in the number field $R(e^{2\pi i/9})$ and corresponding multiplicative equations in the rational field. Here he considers diophantine equations which can be given as multiplicative ideal equations in a suitable form, for instance, as an equality between two norms of linear forms in the field $R(e^{2\pi i/l})$ where l is an odd prime. The fundamental theorem reduces the solving of these equations to the solving of multiplicative equations in the rational domain. In this way the complete solution of some diophantine equations are obtained. For instance, the equation $\sum_{i=1}^{2n} x_i^l$ is solved. The solution of this diophantine equation is reduced to the solution of a system of linear equations of $n(l-2)$ equations in $\frac{1}{2}(n^2-n)+l-1$ unknowns.

H. Bergström (Göteborg).

Samet, P. A. Algebraic integers with two conjugates outside the unit circle. Proc. Cambridge Philos. Soc. **49**, 421–436 (1953).

The following notations are used: S_1 is the set of algebraic integers, ρ, such that all its conjugates, except ρ, satisfy $|z|<1$; S_2 is the set of conjugate algebraic integers ρ and σ such that all their algebraic conjugates, except ρ and σ, satisfy $|z|<1$; S_2' is the subset of real ρ and σ of S_2; S_2'' is the subset of complex ρ and σ of S_2 and thus $\rho=\bar{\sigma}$, T_1 is the set of algebraic integers τ, such that all other algebraic conjugates of τ satisfy $|z|\leqq 1$ and that there actually are some with $|z|=1$; T_2 is the set of algebraic integers which have two algebraic conjugates with $|z|>1$, all others satisfying $|z|\leqq 1$ with some actually on $|z|=1$; T_2' is the subset of real ρ and σ of T_2, T_2'' is the subset of complex ρ and σ of T_2. The sets S_1, S_2, S_2', S_2'' and T_1 have been studied earlier. Salem has proved that S_1 is closed, Kelly that $S_1 \cup S_2''$ is closed. Siegel has shown that S_1 has no limit points in the interval $1\leqq x\leqq 2^{1/2}$ and found all the points of S_1 in this interval. Kelly has shown that $S_1\cup S_2''$ has no limit points on the unit circle. Now the author has found that those numbers of S_2'' which are reciprocal biquadratic units are limit points of S_2''. Further he gives the following criterion for T_2''. Let Θ be a complex number such that $\Theta>1$, $\Theta\neq\bar{\Theta}$. Let μ be a number such that the series $\sum_{n=0}^{\infty}\{\mu\Theta^n+\bar{\mu}\bar{\Theta}^n\}z^n$ has its real part bounded above in $|z|<1$ and does not belong to the Hardy class H^2. Then Θ is an algebraic integer of the set T_2'' and μ an algebraic number of the field $R(\Theta)$. Conversely, if $\Theta \in T_2''$ and $\mu \in R(\Theta)$, this series has its real part bounded above in $|z|<1$ and does not belong to the class H^2. He also gives some results on the distribution of T_2''. For instance, he proves that every number of the set $S_1\cup S_2''$ is a limit point of the set T_2''.

H. Bergström (Göteborg).

***Kantz, Giorgio. Su quelle radici dell'unità di un corpo K ciclico di grado l sopra un corpo algebrico k, le quali sono potenze $(\sigma-1)$-esime di numeri di K, essendo l numero primo e σ un automorfismo generatore di K relativamente a k.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 131–138. Casa Editrice Perrella, Roma, 1953.

Let K be cyclic of prime degree l over the algebraic field k, let σ denote an automorphism of K relative to k and put $\delta=\sigma-1$. The writer characterizes the numbers $\alpha \in K$ such that α^δ is a root of unity.

L. Carlitz (Durham, N. C.).

Leopoldt, Heinrich W. Zur Geschlechtertheorie in abelschen Zahlkörpern. Math. Nachr. **9**, 351–362 (1953).

Suppose that K is a finite abelian extension of the rational field P with the conductor f in the sense of class field theory. This implies that K lies in the field P_f of all fth roots of unity of which one, Z_f, is picked analytically as a primitive root. The Artin isomorphism between the class group of K/P and its Galois group $\mathfrak{G}$ is then given for rational y relatively prime to f as the automorphism (K/y) of K/P which is induced in the sense of the Galois theory by the automorphism $(P_f/y)=(Z_f\to Z_f^r)$ where r is the least positive remainder of y modulo f. Setting $\chi(K/y)=\bar{\chi}(y)$, it follows that each character χ of $\mathfrak{G}$ determines a residue class character modulo f whose conductor shall be denoted by $f(\chi)$. The group $\mathfrak{X}$ of these residue class characters for given K determines K uniquely, and denoting by

$$\tau(\chi)=\sum_{x \bmod f(\chi)} \chi(x)Z^x_{f(\chi)}$$

the normalized Gaussian sum of χ, the author shows the known result that $P_nK=P_n(\tau(\chi),\cdots)$ where n is the exponent of $\mathfrak{X}$ and P_n is the field of all nth roots of unity. After these general preparations it is shown how the class field of genera K^* (Geschlechterklassenkörper) of K, i.e., the maximal abelian extension of P containing K, which is unramified at all finite prime divisors over K, can be determined directly in terms of the arithmetic description of K in P by means of characters. According to the preliminary arguments it suffices to exhibit the character group $\mathfrak{X}^*\supseteq\mathfrak{X}$ of K^*/P. This is achieved, using the decomposition law for primes of the class field theory, by an identification of $\mathfrak{X}^*$ with the direct product of the p-components $\mathfrak{X}_p=\{\chi_p\}$ of $\mathfrak{X}$. The local component of χ_p of χ is determined as usual by the formulas $\chi_p(y)=\chi(y_p)$ if $y_p\equiv y \pmod{p^\nu}$, $y_p\equiv 1 \pmod{f(\chi)/p^\nu}$ with p^ν the p-component of $f(\chi)$. It follows that

$$K^*P_n=P_n(\cdots,\tau(\chi_p),\cdots).$$

Moreover the principal genus of a cyclic extension K/P (the Artin kernel for K^*/K) is identified with the divisor class group which is generated by the $(1-S)$th powers, for all $S\in\mathfrak{G}$, of divisor classes in the restricted sense. Furthermore, other results from the theory of quadratic fields are generalized, for example that the class number (in the restricted sense) of a cyclic field K of prime degree l is precisely then prime to l if the discriminant of K/P is a prime.

O. F. G. Schilling (Chicago, Ill.).

Brandt, Heinrich. Über das quadratische Reziprozitätsgesetz im Körper der dritten Einheitswurzeln. Nova Acta Leopoldina (N.F.) **15**, 163–188 (1952).

The author proves two quadratic reciprocity laws for the field F of the title, both similar to his corresponding laws in the rational and in the Gaussian fields [cf. Comment. Math. Helv. **26**, 42–54 (1952); these Rev. **13**, 726] although one of them differs in an essential feature from its counterparts. The other one states that $[\delta/\kappa]=[\kappa/\delta]$, where δ is a "proper" or "improper" prime-discriminant of binary quadratic forms over F, κ is either a unit of F or a prime of F prime to δ, and the symbols have the following meanings. The proper values of δ are -4, 8, -8, and all primes π of F such that $\pi\equiv 1 \pmod 4$. They are actual discriminants, as are also their products by ρ^2 and ρ^4, $\rho=\exp(2\pi i/6)$. The improper values of δ are 4ω, -4ω, 8ω, -8ω, $\bar{\pi}\omega$, where $\bar{\pi}$ is any prime of F such that $\bar{\pi}\equiv 1+2\rho \pmod 4$, and ω is a symbolic factor whose square is to be replaced by 1 in products of prime-discriminants forming ordinary discriminants. The symbol $[\kappa/\delta]$ is the character $\chi_\delta(\kappa)$ defined by:

(1) when $\delta=\pi$, $\chi_\delta(\kappa)=1$ or -1 according as κ is a quadratic residue or non-residue mod π; (2) when $\delta=\bar{\pi}\omega$, $\chi_\delta(\kappa)=1$ or -1 according as κ is a quadratic residue or non-residue mod $\bar{\pi}$; (3) the values $\delta=-4, 8, -8, 4\omega, -4\omega, 8\omega, -8\omega$, are related to the seven distinct subgroups of index 2 of the group of primitive residues in F modulo 8, and $\chi_\delta(\kappa)=1$ or -1 according as κ is in the related subgroup or its coset. The symbol $[\delta/\kappa]$ is defined by: (4) when δ is proper, $[\delta/\kappa]=1$ or -1 according as κ is or is not represented by an integral binary quadratic form over F of discriminant δ; (5) when $\delta=\bar{\pi}\omega$, $[\delta/\kappa]=[\bar{\pi}/\kappa]=1$ or -1 according as $\bar{\pi}$ is or is not a quadratic residue modulo κ; (6) when $\delta=\pm 2^r\omega$, $[\delta/\kappa]$ is defined to be 1 or -1 according as $\pm 2^r$ is a quadratic residue or non-residue modulo κ. *R. Hull.*

Brandt, Heinrich. Über die Reduktion der positiven ternären quadratischen Formen. Math. Nachr. 9, 249–254 (1953).

Among the forms equivalent to a real positive definite ternary quadratic form

$$f=a_1x_1^2+a_2x_2^2+a_3x_3^2+a_{23}x_2x_3+a_{13}x_1x_3+a_{12}x_1x_2,$$

that is, forms obtained from f by integral unimodular transformations on the x's, of determinant 1, those for which

$$0<a_1\leqq a_2\leqq a_3,\quad |a_{12}|\leqq a_1,\quad |a_{13}|\leqq a_1,\quad |a_{23}|\leqq a_2,$$
$$|a_{23}|+|a_{13}|+|a_{12}|\leqq a_1+a_2$$

if $a_{23}a_{13}a_{12}\leqq 0$, are called semi-reduced. Among the semi-reduced forms equivalent to f, there are forms for which further $|a_{23}|$, $|a_{13}|$, $|a_{12}|$ have values m_1, m_2, m_3 in some order, where m_1 is an absolute minimum, m_2 is a minimum for the other two when one is m_1, and m_3 is a minimum for the other one when two of them are m_1 and m_2, resp. If two or more permutations of m_1, m_2, and m_3, occur, retain only the forms with lexicographic precedence. At most a change of sign of two of the variables will produce the further conditions $a_{23}\geqq 0$, $a_{13}\geqq 0$, $a_{12}\geqq 0$, unless all three are not 0. Then a unique reduced form is obtained except for the possibility of $a_{23}\leqq 0$, $a_{13}>0$, $a_{12}>0$. The foregoing conditions have recently been employed by O. Intrau [dissertation as yet unpublished] to construct tables of integral reduced forms, listed by genus and order, and to derive simple rules for determining the number of proper automorphs of such forms. *R. Hull* (Lafayette, Ind.).

***Rényi, Kató. The distribution of numbers not divisible by a kth power of an integer greater than one in the set of values of a polynomial having rational roots.** Comptes Rendus du Premier Congrès des Mathèmaticiens Hongrois, 27 Août–2 Septembre 1950, pp. 493–506. Akadémiai Kiadó, Budapest, 1952. (Hungarian. Russian summary)

Let $P(x)$ be a polynomial of degree r whose roots are all rational. Then the density of integers n for which $P(n)$ is kth power free exists and is positive except if $P(x)$ has a k-fold root or if $P(x)$ is such that there exists a p so that, for every n, $P(n)\equiv 0 \pmod{p^k}$. Clearly the density in question is 0 in the exceptional cases. The second exceptional case is discussed in detail. The proofs are elementary. *P. Erdös* (South Bend, Ind.).

***Szüsz, Péter. Über ein Problem der Gleichverteilung.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 461–472. Akadémiai Kiadó, Budapest, 1952. (Hungarian. Russian and German summaries)

The author proves the following theorem: Let

$$P_\nu=\{(x_\nu^{(1)}, x_\nu^{(2)}, \cdots, x_\nu^{(p)})\}$$

be a sequence of points in p-dimensional space. Assume that

$$\sum_{\nu=1}^{n} \exp\left[i(k_1x_\nu^{(1)}+k_2x_\nu^{(2)}+\cdots+k_px_\nu^{(p)})\right]\leqq\psi(k_1, k_2, \cdots, k_p).$$

Let Y be a p-dimensional rectangle, i.e., $y=(y_1, y_2, \cdots, y_p)$ is in Y if $\alpha_i\leqq y_i\leqq\beta_i$ for $i=1, 2, \cdots, p$. We further assume $0\leqq\alpha_i<\beta_i\leqq 2\pi$, $i=1, 2, \cdots, p$. Let $N(n, Y)$ denote the number of P_ν, $1\leqq\nu\leqq n$, whose coordinates reduced mod 2π are in Y, and finally let m be any integer. We then have

$$\left|N(n, Y)-\frac{M(Y)}{(2\pi)^p}n\right|$$
$$<k_p\left(\frac{n}{m+1}+\sum_{k_1=-m}^{m}{}' \sum_{k_2=-m}^{m}{}' \cdots \sum_{k_p=-m}^{m}{}' \frac{\psi(k_1, k_2, \cdots, k_p)}{(|k_1|+1)\cdots(|k_p|+1)}\right)$$

where $M(Y)=\prod_{i=1}^p(\beta_i-\alpha_i)$ is the volume of Y, k_p depends only on p and the dash in the summation sign indicates that the term $k_1=k_2=\cdots=k_p=0$ should be omitted.

For $p=1$ this result was proved by the reviewer and Turán [Nederl. Akad. Wetensch., Proc. 51, 1146–1154, 1262–1269 (1948); these Rev. 10, 372], and the author's proof is similar to ours. The author quotes a paper by Koksma [Math. Centrum Amsterdam, Scriptum no. 5 (1950); these Rev. 12, 394] who discovered the same result simultaneously with the author. *P. Erdös.*

Korobov, N. M. Unimprovable estimates of trigonometric sums with exponential functions. Doklady Akad. Nauk SSSR (N.S.) 89, 597–600 (1953). (Russian)

In an earlier paper [Trudy Mat. Inst. Steklov. 38, 87–96 (1951); these Rev. 14, 143] the author showed that, in a certain sense, for most values of the real number α the inequality

$$|S|=\left|\sum_{x=1}^{P}\exp\,(2\pi im\alpha q^x)\right|<cP^{\frac{1}{2}}$$

holds, where c depends only upon the positive integers m and $q>1$. He now shows that this upper bound $cP^{\frac{1}{2}}$ can be replaced by an arbitrarily slowly increasing function $\phi(P)$ tending to infinity with P, for a certain class of values of α. In fact, if $\phi(P)$ is such a function, integers n_ν ($\nu=0, 1, 2, \cdots$) can be chosen sufficiently large such that for

$$\alpha=\sum_{\nu=0}^{\infty}\frac{1}{m_{\nu+1}q^{n_\nu}}$$

we have $|S|=o\{\phi(P)\}$. Here $\{m_\nu\}$ is a sequence of integers greater than unity and prime to q defined recursively by $m_{\nu+1}=m_\nu(q^{\tau_\nu}-1)$ ($\nu=1, 2, 3, \cdots$) where $\{\tau_\nu\}$ is an arbitrary strictly increasing sequence of positive integers such that τ_ν is a multiple of the exponent of m_ν modulo q_ν. The integers n_ν are then chosen to satisfy $n_\nu>\phi_1(\nu m_\nu\tau_\nu)$ where ϕ_1 is the inverse function to ϕ. The result is best possible in the sense that $o\{\phi(P)\}$ cannot be replaced by $O(1)$ for every α of the class considered. *R. A. Rankin* (Birmingham).

Volkmann, Bodo. Über Klassen von Mengen natürlicher Zahlen. J. Reine Angew. Math. 190, 199–230 (1952).

Let A be a sequence of positive integers. Put $f(A)=\sum 1/2^{a_i}$ where $a_1<a_2<\cdots$ are the elements of A. The sequence A is called rational if $f(A)$ is a rational number. The author proves that if A and B are rational, then their Schnirelmann sum is also rational. A sequence A is called pseudorational (this concept is due to Buck) if to every ϵ there exist rational sequences B and C, $B\subset A\subset C$, so that the density of $C-B$ is less than ϵ. Various properties of pseudorational sequences are proved.

Finally the author studies the Hausdorff dimension of various classes of sets. He proves amongst others the following theorems: 1) Let A be a sequence of integers. Let ϕ_A be the set of real numbers which corresponds to the infinite subsequences of A. Then the Hausdorff dimension of ϕ_A equals the lower density of A. 2) The Hausdorff dimension of the set whose elements correspond to sequences which do not contain i consecutive integers equals $\log \gamma_i/\log 2$ ($i \geqq 2$) where γ_i is the greatest real root of $x^i = x^{i-1} + x^{i-2} + \cdots + 1$.

P. Erdös (South Bend, Ind.).

ANALYSIS

Green, John W. Support, convergence, and differentiability properties of generalized convex functions. Proc. Amer. Math. Soc. **4**, 391–396 (1953).

Let $\mathfrak{F}$ be a family of continuous real functions defined in an interval $I=(a, b)$ with the property that if $a<x_1<x_2<b$, then there is a unique $F \in \mathfrak{F}$ taking arbitrary values at x_1, x_2. The sub-F functions introduced by Beckenbach [Bull. Amer. Math. Soc. **43**, 363–371 (1937)] are those functions f such that if $a<x_1<x_2<b$ and G is the member of $\mathfrak{F}$ coinciding with f at x_1 and x_2 then $f(x) \leqq G(x)$ in (x_1, x_2). The author defines sub-FS functions to be the sub-F functions associated with a class $\mathfrak{F}$ which has the additional properties: (A) each $F \in \mathfrak{F}$ is continuously differentiable; (B) if $F_n \rightarrow F$, then $F_n' \rightarrow F'$ uniformly in each compact subinterval of I. Among other results, it is proved that a sub-FS function has a derivative except perhaps at a countable set of points. If f_n, $n=1, 2, \cdots$, are sub-F functions and $f_n \rightarrow f$, then f is sub-F and the convergence is uniform on compact subintervals of I. If the f_n are sub-FS then, apart from a countable set of points, $f_n' \rightarrow f'$ boundedly on compact subintervals of I. [The reviewer wishes to draw attention to an error in his own paper on this subject [Quart. J. Math., Oxford Ser. (2) **1**, 100–111 (1950); these Rev. **12**, 83]. The assertion in Theorem 3, Corollary (ii), repeated in Theorem **4**, that f has a second derivative p.p. is false. In fact Corollary (ii) does not follow from Corollary (i). The present author quotes, but does not use, this assertion.]

F. F. Bonsall (Newcastle-upon-Tyne).

Herrera, Félix E. A note on differentiation of arbitrary real order. Univ. Nac. Tucumán. Revista A. **9**, 79–85 (1952). (Spanish)

By repeated integration by parts, the author expands a Riemann-Liouville fractional derivative into a form analogous to Taylor's series with remainder, having ordinary derivatives as coefficients. *R. P. Boas, Jr.*

Whittaker, J. M. A two-point boundary problem for infinitely differentiable functions. Quart. J. Math., Oxford Ser. (2) **4**, 136–141 (1953).

The author has called a pair of sequences $(p; q)$ complete if the operators $f^{(p_n)}(1)$, $f^{(q_n)}(0)$ form a basic set, and proposed the question [cf. his Interpolatory function theory, Cambridge University Press, 1935] of whether, corresponding to any two entire functions f, g, it is always possible to find an entire function h such that $h^{(p_n)}(1) = f^{(p_n)}(1)$, $h^{(q_n)}(0) = g^{(q_n)}(0)$. This question has been answered only in special cases, but here the author shows that it can be answered completely if functions of class C^∞ replace entire functions: if $(p; q)$ is complete and A_n, B_n are arbitrary numbers, the equations $f^{(p_n)}(1) = A_n$, $f^{(q_n)}(0) = B_n$ ($n \geqq 1$) have a solution in C^∞. Similarly $f^{(n)}(a_n) = A_n$ has a solution in C^∞. The author proposes the problem of whether the equation $g(x+1) - g(x) = f(x)$ always has a solution in C^∞ when $f(x)$ is in C^∞; the answer is affirmative for entire functions [Guichard; it is equivalent to the solution of the two-point problem for $(p; q) = (2n; 2n)$; cf. the reference above].

R. P. Boas, Jr. (Evanston, Ill.).

Schoenberg, I. J. On smoothing operations and their generating functions. Bull. Amer. Math. Soc. **59**, 199–230 (1953).

This is an exposition of various problems most of which were solved by the author and his collaborators, grouped under the general title of smoothing operations. The problems deal mainly with the characterization and properties of certain transformations belonging to one of the following types. (*) The finite linear transformation $y_i = \sum_1^n a_{ik} x_k$, $i=1, \cdots, m$. (**) The sequence convolution $y_n = \sum_{-\infty}^{\infty} a_{n-\nu} x_\nu$. (***) The integral convolution $g(x) = \int_{-\infty}^{\infty} \Lambda(x-t) f(t) dt$. A problem considered for the three types is the characterization of variation-diminishing transformations. For (*) this was solved (partially) by the author [Math. Z. **32**, 321–328 (1930)] and by Motzkin [Basel dissertation, 1933, Jerusalem, 1936]. A different solution was also given by Schoenberg and Whitney [Compositio Math. **9**, 141–160 (1951); these Rev. **13**, 98] who also considered cyclic v.d.t. For (**) the problem was solved by the author [Courant Anniversary Volume, Interscience, New York, 1948, pp. 351–370; these Rev. **9**, 337]; the (necessary and sufficient) condition being that $\{a_n\}$ is a totally positive normalized sequence. For (***) the problem was solved by the author [Acta Sci. Math. Szeged **12**, Pars B, 97–106 (1950); these Rev. **12**, 23]. A necessary and sufficient condition in this case is that except for a sign $\Lambda(x)$ be a Pólya frequency function.

Various other problems are discussed: notably, when can a transformation (**) be referred to as a smoothing operation [Schoenberg, Quart. Appl. Math. **4**, 45–99, 112–141 (1946); these Rev. **7**, 487; **8**, 55; and Courant Anniversary Volume, same ref. as above]; convex curves in higher dimensional spaces, and the problem of characterizing totally positive sequences by means of the associated generating function $\sum_{-\infty}^{\infty} a_n z^n$ which was successfully solved by the author [Courant Anniversary Volume, same ref.], Aissen, Schoenberg and Whitney [J. Analyse Math. **2**, 93–103 (1952); these Rev. **14**, 732] and Edrei [ibid. **2**, 104–109 (1952); Canadian J. Math. **5**, 86–94 (1953); these Rev. **14**, 732].

S. Agmon (Jerusalem).

***Fejér, Lipót. Approximation durch Interpolation.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 99–112. Akadémiai Kiadó, Budapest, 1952. (Hungarian. Russian and German summaries)

In the first two chapters the author discusses the history of interpolation and mechanical quadratures. Then he gives an outline of some of his own results. The paper is expository in character and is a very readable and interesting introduction to the subject. *P. Erdös* (South Bend, Ind.).

***Huszár, Géza. Sur une méthode nouvelle d'interpolation.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 727–730. Akadémiai Kiadó, Budapest, 1952. (Hungarian. Russian and French summaries)

If $L(y_i)$ is Lagrange's interpolation formula for a function whose values at the points x_i are y_i, the author's interpola-

tion formula may be written $[L(y_i^{m+n})/L(y_i^n)]^{1/m}$. The author states that in many cases an adroit choice of the exponents m and n results in a tremendous accuracy.

A. Erdélyi (Pasadena, Calif.).

Gagua, M. B. On representation of functions by series of particular solutions of elliptic equations. Soobščeniya Akad. Nauk Gruzin. SSR **13**, 321–327 (1952). (Russian)

For the differential equation

$$(1)\qquad \Delta u+a(x,y)u_x+b(x,y)u_y+c(x,y)u=0$$

with a, b, c entire, the author considers the special solutions

$$(2)\qquad u_n(x,y)=\mathrm{Re}\left[G(z,0,z,\bar z)P_n(z)-\int_0^z P_n(t)\frac{\partial}{\partial t}G(t,0,z,\bar z)dt\right].$$

Here $G(t,\tau,z,\zeta)$ is the complex Riemann's function for (1), $P_n(z)$ is an arbitrary polynomial of degree n, and $z=x+iy$. In complete analogy to the problem of the uniform approximation of functions of a complex variable by polynomials, the author studies the uniform approximation of function $f(x,y)$ by means of the special solutions (2). Let E be closed and bounded. A necessary and sufficient condition is derived in order that an arbitrary function which is continuous on E and satisfies (1) in the interior of E be uniformly approximable by solutions (2). This condition is automatically fulfilled if E is a closed simply connected region for which the Dirichlet problem for (1) is stable. It follows also that if E is a closed bounded nowhere dense set which does not separate the plane, then any $f(x,y)$ which is continuous on E can be uniformly approximated on E by the functions (2). This is the analog of a theorem of Walsh-Lavrentieff for polynomials.

P. Davis (Washington, D. C.).

Theory of Sets, Theory of Functions of Real Variables

Riguet, Jacques. Systèmes de coordonnées relationnels. C. R. Acad. Sci. Paris **236**, 2369–2371 (1953).

The author considers general properties of finite sets R of permutable equivalence relations on a set E. He shows that the usual formal properties of axes, hyperplanes, etc., are valid for such more general "coordinate systems".

G. Birkhoff (Cambridge, Mass.).

***Markwald, Werner. Zur Theorie der konstruktiven Wohlordnungen.** Dissertationen der mathematisch-naturwissenschaftlichen Fakultät der westfälischen Wilhelms-Universität zu Münster in Referaten, Heft 2, pp. 10–12. Aschendorffsche Verlagsbuchhandlung, Münster, 1952.

Kurepa, Djuro. Sur la relation d'inclusion et l'axiome de choix de Zermelo. Bull. Soc. Math. France **80**, 225–232 (1952).

Let a and b be any sets. We write: $K(a,b)$, if $a\subset b$ or $b\subset a$; $D(a,b)$, if $a\cap b=0$; $I(a,b)$, if $a\cap b'\neq 0$ and $a'\cap b\neq 0$; let K', D', I' be the negations of K, D, and I, respectively. Let R be any of the above-mentioned predicates and let F be any family of sets. We write $R(F)$, if F contains a maximal subfamily F_0 such that $R(a,b)$ whenever a and b are distinct elements of F_0. It has been proved by Birkhoff and Vaught, respectively, that the statements $(F)K(F)$ and $(F)D(F)$ are equivalent to the axiom of choice, Z. Kurepa now proves that this holds also for the statements $(F)I(F)$, $(F)D'(F)$, $(F)I'(F)$, as well as for the conjunction of the statement $(F)K'(F)$ with the ordering principle (it was shown by Mostowski that the ordering principle separately does not imply Z). Moreover, the following interesting problem is raised. We can consider such statements as $(F)[K(F)\vee D'(F)]$, $(F)[K(F)\vee D(F)\vee I(F)]$, each of which follows from Z. Is Z implied by any of these statements?

E. W. Beth (Amsterdam).

Novák, Josef. On partition of an ordered continuum. Fund. Math. **39** (1952), 53–64 (1953).

Let C be a continuous ordered set, and $\mathfrak{m}(C)$ be the smallest cardinal number such that there exists a subset of C of power $\mathfrak{m}(C)$ which is dense in C. A set $\mathfrak{P}$ of intervals (each of which is closed and contains more than one point) of C is called a (dyadic) partition [cf. the author's paper, Fund. Math. **37**, 77–83 (1950); these Rev. **13**, 330] of C if the following conditions are satisfied. (a) If $X\in\mathfrak{P}$ and $Y\in\mathfrak{P}$, then either $X\cap Y=X$ or $X\cap Y=Y$, or $X\cap Y$ is not an interval. (b) $C\in\mathfrak{P}$. (c) If $X\in\mathfrak{P}$, then there exist $X_1\in\mathfrak{P}$ and $X_2\in\mathfrak{P}$ such that $X=X_1\cup X_2$ and $X_1\cap X_2$ is a single element d of C. (d) The intersection of every decreasing set of intervals belonging to $\mathfrak{P}$ is either an interval belonging to $\mathfrak{P}$ or a single element of C. It is shown that every continuous ordered set C has at least one partition $\mathfrak{P}$, and the power of every partition $\mathfrak{P}$ of C is $\mathfrak{m}(C)$. For a given partition $\mathfrak{P}$ of C and a given $X\in\mathfrak{P}$, the element d described in (c) is unique; if D denotes the set of all elements d obtained by letting X vary in $\mathfrak{P}$, then D is dense in C and $|D|=\mathfrak{m}(C)$. The order $\alpha(I)$ of an interval I of C is defined as the order type of the set of intervals belonging to $\mathfrak{P}$ and containing I, ordered inversely with respect to inclusion; $\alpha(I)$ exists and is a positive ordinal number. The order $\alpha(\mathfrak{P})$ of $\mathfrak{P}$ is defined as the smallest ordinal such that there is no $X\in\mathfrak{P}$ of order $\alpha(X)>\alpha(\mathfrak{P})$. If C is a continuous ordered set, then there exists an ordinal μ of power $\leqq\mathfrak{m}(C)$ such that C is similar to a lexicographically ordered set whose elements are transfinite sequences, of order types $\leqq\mu$, of zeros and ones; the question whether μ may be taken to be the initial number of $Z(\mathfrak{m}(C))$ is answered in the affirmative by Novotný [cf. (2) of the following review]. The author shows that a necessary and sufficient condition that every continuous ordered set of power $\aleph_\nu$ contain at least one element of character c_{00} is that $\aleph_\nu<2^{\aleph_1}$ [the sufficiency, however, is due to Hausdorff, Ber. Verh. Sächs. Ges. Wiss. Leipzig. Math.-Phys. Kl. **59**, 84–159 (1907), p. 135]. For every continuous ordered set C with $|C|=2^{\aleph_0}$ and $\mathfrak{m}(C)\leqq\aleph_1$, the set A_{00} of all elements of C having character c_{00} is not empty (this answers a question raised by the author [Czechoslovak Math. J. **1**(76), 63–79 (1951), p. 79; these Rev. **14**, 146]); and in case $\mathfrak{m}(C)=\aleph_1$, A_{00} is dense in C and of power $2^{\aleph_0}$.

F. Bagemihl (Princeton, N. J.).

Novotný, Miroslav. Sur la représentation des ensembles ordonnés. Fund. Math. **39** (1952), 97–102 (1953).

[Cf. the preceding review.] (1) If C is a continuous ordered set with $\mathfrak{m}(C)=\aleph_\nu$, then there exists a partition $\mathfrak{P}$ of C with $\alpha(\mathfrak{P})\leqq\omega_\nu$. (2) If M is an ordered set of at least two elements, let $\mathfrak{s}(M)$ be the smallest cardinal number such that there exists a subset H of M of power $\mathfrak{s}(M)$ which is dense in M in the following sense [Hausdorff, Grundzüge der Mengenlehre, Veit, Leipzig, 1914, p. 89]: if $a\in M$, $b\in M$, and $a<b$, then there exist $a'\in H$, $b'\in H$, such that $a\leqq a'<b'\leqq b$. If $\mathfrak{s}(M)=\aleph_\nu$, then M is similar to a lexico-

graphically ordered set whose elements are transfinite sequences, of order type ω_ν, of zeros and ones. (3) Novák [Czechoslovak Math. J. 1(76), 63–79 (1951), p. 64; these Rev. 14, 146] has proved that if C is a continuous ordered set, and if $\mathfrak{C}$ is a class of at least two mutually exclusive subsets of C whose union is C, each of which is an interval or a single point of C, then $\mathfrak{C}$, ordered in the natural way, is continuous. The author shows that every continuous ordered set C with $\mathfrak{m}(C)=\aleph_\nu$ is similar to a naturally ordered class of mutually exclusive subsets of a certain continuous ordered set V_{ω_ν}, each of which is an interval or a single point of V_{ω_ν}, and whose union is V_{ω_ν}. *F. Bagemihl.*

Ginsburg, Seymour. A cardinal number associated with a family of sets. Proc. Amer. Math. Soc. 4, 573–577 (1953).

For each set family U the author defines the "maximal density" md (U) and the "containing maximal density" cmd (U) in the following way: md $(U)=\inf_A p(\bigcup_x x)$ ($x \in A$, A is a maximal subfamily of U consisting of pairwise inclusionally incomparable elements; p=power); cmd $(U)=\inf_V$ md (V) ($V\subseteq U$, V is coinitial with the ordered set $(U;\subseteq)$). If the ordered set $(U;\supseteq)$ is ramified, then md (U)=cmd (U) (Theorem 1). The main result (Theorem 2) affirms the commutativity of the operator cmd and the cartesian multiplication: for any family system U^ξ $(\xi<\alpha)$ one has cmd $(\prod_\xi U^\xi)=\prod_\xi$ cmd (U^ξ) $(\xi<\alpha)$. The proof is based upon the existence of some p-homogeneous subfamily of U which is coinitial with $(U;\subseteq)$ (Lemma 2). (A family F is p-homogeneous, provided for each $X\in F$ and each $Y\in F$ with $Y\subseteq X$ one has $pX=pY$.) *G. Kurepa.*

Jones, F. Burton. On certain well-ordered monotone collections of sets. J. Elisha Mitchell Sci. Soc. 69, 30–34 (1953).

This paper deals with an infinity lemma and reproves a proposition contained in the reviewer's thesis [Paris, 1935 = Publ. Math. Univ. Belgrade 4, 1–138 (1935); especially p. 76, Th. 2, p. 78, Th. 4, p. 80, Th. 5, §9, pp. 89–98]. The author gives a proof of the existence of what the reviewer calls the Aronszajn ramified table [≡ordered set S so that for each $x\in S$ the set $(\cdot,x)_S$ of all predecessors of x is a well ordered chain and that moreover S is non-denumerable and that each chain as well as each row of S is $\leq\aleph_0$; the α-row of S is the set of all the $x\in S$ such that the order type of $(\cdot,x)_S$ equals α]. The existence of such an S is due to Aronszajn and Aronszajn's set was published in the reviewer's thesis [p. 96; cf. also Publ. Math. Univ. Belgrade 6–7, 129–160 (1938)] the author points ~~where it is pointed~~ out that he possessed the same proof in 1933, when he tried to settle the still open problem of finding whether each normal space satisfying the Moore axioms 0 and 1 is metric [cf. R. L. Moore, Foundations of point set theory, Amer. Math. Soc. Colloq. Publ. v. 13, New York, 1932]. *G. Kurepa.*

Eyraud, Henri. Ensembles agrégatifs adjoints. Ann. Univ. Lyon. Sect. A. (3) 15, 5–16 (1952).

The purpose of this note is to rectify an error in an earlier paper by the same author [same Ann. (3) 14, 5–28 (1951), p. 12, Theorem 3, II; these Rev. 14, 255]. *F. Bagemihl.*

Errera, Alfred. Le problème du continu. Atti Accad. Ligure 9 (1952), 176–183 (1953).

Remarks aimed to show the plausibility of taking the continuum hypothesis or its denial as a new postulate in set theory. *F. Bagemihl* (Princeton, N. J.).

Moneta, J. Application du théorème du continu. Cahiers Rhodaniens 4, 29–42 (1952).

It is shown that in the plane there exists a point set which intersects every straight line with a rational slope in precisely one point, and every straight line with an irrational slope in precisely two points; an analogous result is obtained for three-dimensional space. [Cf. Rosenthal, S.-B. Math.-Phys. Kl. Bayer. Akad. Wiss. 1922, 221–240, esp. pp. 223–224, 226–227.] The author appeals to the continuum hypothesis (which he calls the "continuum theorem"), but an examination of the proof shows that this hypothesis is unnecessary. *F. Bagemihl* (Princeton, N. J.).

Padmavally, K. Generalization of rational numbers. Revista Mat. Hisp.-Amer. (4) 12, 249–265 (1952).

Let $[\tau]$ denote the type obtained from an order type τ by filling its gaps and supplying its missing border elements. A union of order types τ_x $(x\in X)$ is the order type of an ordered set which is a union of ordered sets T_x $(x\in X)$, having the respective order types τ_x, where the relative order of the elements of each T_x is unaltered and superposition of elements is allowed. Denote by η the order type of the set of rational numbers, and by θ the order type of the closed unit interval of real numbers. The familiar results that η is dense in θ and is similar to an isolated subset of θ, that every enumerable order type can be embedded in η, and that θ can be embedded in every one of its intervals (where an interval of an ordered set is taken to mean the subset of all elements between two distinct elements of that set), are extended to certain complete powers $\tau((\alpha))$ [see Hausdorff, Math. Ann. 65, 435–505 (1908), p. 460], where τ denotes an order type $(|\tau|\geqq 2)$ and $\alpha>1$ an ordinal number, as follows. (1) If α is a limit number, then $[\tau((\alpha))]$ has a dense subset which is similar to an isolated subset of $[\tau((\alpha))]$, and if, furthermore, τ has a first or a last element, then $\tau((\alpha))$ has a dense subset which is similar to an isolated subset of $\tau((\alpha))$. (2) If ω_β is a regular initial number, then any union of $\aleph_\beta$ order types, each of which is similar to an isolated subset of $[\tau]((\omega_\beta))$, is similar to an isolated subset of $[\tau]((\omega_\beta))$. (3) If α is indecomposable, then $\tau((\alpha))$ can be embedded in every one of its nonempty intervals. The author also shows that a necessary condition for an order property L to be the minimal falling and iterative supertype of a noniterative property [see Gleyzal, Trans. Amer. Math. Soc. 48, 451–466 (1940); these Rev. 2, 129] is that there exist an enumerable set of order types each having property L, and a union of these types which does not have property L. The paper contains many misprints. [Theorem V on p. 253, which the author ascribes to Webber, Trans. Roy. Soc. Canada. Sect. III, 25, 65–74 (1931), was proved by Hausdorff, Ber. Verh. Sächs. Ges. Wiss. Leipzig. Math.-Phys. Kl. 58, 106–109 (1906), p. 143.] *F. Bagemihl.*

Eggleston, H. G. A correction to a paper on the dimension of cartesian product sets. Proc. Cambridge Philos. Soc. 49, 437–440 (1953).

The author acknowledges the incorrectness of his proof that $\dim A\times B\geqq\dim A+\dim B$, where $\dim A$ is the Besicovitch dimension of the arbitrary set A [same Proc. 46, 383–386 (1950); these Rev. 12, 323]. Partial results which remain valid are considered, and it is shown by a different method that the inequality holds if the β-dimensional Hausdorff measure of B is infinite for some β and A includes a closed subset of positive α-measure. *L. H. Loomis.*

Pucci, Carlo. Sulla compattezza di successioni di funzioni reali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **14**, 471–476 (1953).

Suppose f_n is a sequence of real-valued functions, the domain of each f_n being a subset A_n of Euclidean k-space. Suppose the sequence A_n is uniformly bounded and converges to a set A. The author states: (1) If f_n is pseudoequicontinuous and uniformly bounded, then some subsequence of f_n is biconvergent to a continuous function on A. (2) If f_n is biconvergent, then f_n is pseudoequicontinuous and is eventually uniformly bounded. (Here f_n is said to be "pseudoequicontinuous" provided for each $\epsilon>0$ there exist $n_\epsilon>0$ and $\delta_\epsilon>0$ such that $|f_n(x)-f_n(x')|<\epsilon$ whenever $x, x'\in A_n$, $n>n_\epsilon$, and $|x-x'|<\delta_\epsilon$; and f_n is "biconvergent" provided $f_n(x_n)$ is convergent whenever $x_i\in A_i$ for each i and x_n is convergent.) When the A_n's are all equal to A, (1) reduces to the well-known theorem of Arzelà [Mem. Accad. Sci. Ist. Bologna (4) **3**, 117–141 (1906)]. The present paper includes a proof of (1), and states that (2) will be proved in a forthcoming publication. With the aid of (1), it is proved that under certain circumstances, some subsequence of f_n is biconvergent to a continuously differentiable function on A. *V. L. Klee* (Seattle, Wash.).

Puig Adam, P. Sur les limites de certaines fonctions de partition. Revista Mat. Hisp.-Amer. (4) **13**, 92–101 (1953).

The author considers a real-valued function $f(D)$ from the partitions D: $a=x_0<x_1<\cdots<x_n=b$ of the interval $[a, b]$ such that, for some positive number δ, $f(D)<f(D')$ $[f(D)>f(D')]$ provided D has mesh less than δ and D' is a refinement of D. Under a certain continuity condition, $f(D)$ has a limit as the mesh of D approaches 0. Several examples are considered. *H. S. Wall* (Austin, Tex.).

Puig Adam, P. On the limits of certain partition functions. Revista Mat. Hisp.-Amer. (4) **13**, 102–111 (1953). (Spanish)

Spanish version of the paper reviewed above.

Shukla, U. K. On points of non-symmetrical differentiability of a continuous function. I. Ganita **2**, 54–61 (1951).

Let $f(x)$ be a continuous real function on the real interval (a, b), and let x $(a<x<b)$ be a point where $f(x)$ has finite derivative A. According as x is, or is not, a limit point of the set of points $x+h$ ($h>0$ where $\{f(x+h)-f(x)\}/h=A$) x is termed a point D_{1+}, or D_{0+}, of $f(x)$. Points D_{1-} and D_{0-} are defined analogously. A point which is either D_{1+} and D_{0-} or else D_{0+} and D_{1-} is said to be a point of non-symmetric differentiability of $f(x)$. It is shown that there exists a function, differentiable almost everywhere in its interval of definition, whose points of non-symmetric differentiability form a set of positive measure. *T. A. Botts.*

Nakamura, Masahiro. Uniform space having volume. Math. Japonicae **2**, 193–194 (1952).

The author shows that if a uniform structure S carries a measure μ which is invariant, in the sense that $\mu(V_\alpha(x))=\mu(V_\alpha(y))$ for every index α of the structure and any two points x and y, then S is locally totally bounded. This generalizes a result of Weil on groups with invariant measures [L'integration dans les groupes topologiques et ses applications, Hermann, Paris, 1940; these Rev. **3**, 198]. *L. H. Loomis* (Cambridge, Mass.).

★**Ottaviani, Giuseppe. Sull'integrale stocastico.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 610–616. Casa Editrice Perrella, Roma, 1953.

The author discusses the definition of the Stieltjes integral $\int_a^b X(u)dY(u)$ as the limit of sums in which the points of division of $[a, b]$ are chosen at random. The point of the paper is apparently to give the reader insight into such questions by interpreting random variables as measurable functions on the interval $[0, 1]$. *J. L. Doob.*

★**McShane, Edward J. Order-preserving maps and integration processes.** Annals of Mathematics Studies, no. 31. Princeton University Press, Princeton, N. J., 1953. vi+136 pp. $2.75.

Depuis la définition de l'intégrale donnée par Daniell [Ann. of Math. (2) **19**, 279–294 (1918)], et le développement de la théorie des ensembles ordonnés, il a été clair qu'il était possible de généraliser le procédé d'extension de Daniell au cas où l'"intégrale" prend ses valeurs dans un ensemble ordonné convenable (au lieu de l'ensemble des nombres réels). C'est ce programme que réalise l'auteur, en s'attachant à donner à ses énoncés la plus grande généralité possible, ce qui lui permet d'englober les résultats de ses prédécesseurs (Stone, Nakano, Bochner) et d'élargir le champ des applications de sa théorie. Sous sa forme générale, le problème est le suivant: étant donnés deux ensembles ordonnés F, G, une partie E de F et une application croissante I_0 de E dans G, il s'agit de prolonger I_0 en une application croissante dans G d'une partie de F contenant E et ayant des propriétés convenables relatives à l'ordre et à la structure algébrique éventuelle de G; dans le cas classique, E est l'ensemble des "fonctions élémentaires" et il s'agit d'étendre l'"intégrale" I_0 aux "fonctions sommables". Les postulats additionnels sur F, G, E et I_0 varient suivant les théorèmes démontrés. Pour les plus importants d'entre eux, F est un ensemble réticulé où

$$\sup(x, \inf_\alpha (y_\alpha)) = \inf_\alpha (\sup(x, y_\alpha))$$

et la loi analogue en permutant inf et sup (distributivité infinie); en outre, toute partie dénombrable de F a une borne inférieure et une borne supérieure (dans certains énoncés, cela est supposé vrai de toute partie de F); G est un groupe ordonné où toute partie filtrante et majorée a une borne supérieure. L'axiome de Daniell est introduit sous une forme plus faible: lorsque $\sup S_1$ et $\inf S_2$ existent dans F (S_1 et S_2 parties de E) et $\sup S_1 \geqq \inf S_2$, on doit avoir $\sup I_0(S_1) \geqq \inf I_0(S_2)$ si les deux membres existent dans G, mais cela n'est supposé que pour des ensembles S_1, S_2 qui sont filtrants (croissant et décroissant respectivement) pour une relation d'ordre $\ll$ "plus forte" que $\leqq$ (nous renvoyons à l'ouvrage (chap. I) pour une définition précise de cette notion; par exemple, pour des fonctions, $f\ll g$ peut vouloir dire qu'en tout point x, il y a un voisinage V de x et une constante c tel que $f(y)\leqq c\leqq g(y)$ dans V); l'auteur fait un grand usage de cette notion dans tout le livre. L'ensemble E n'est pas nécessairement supposé réticulé; cette condition est remplacée par des conditions plus faibles, ne portant que sur des couples (e_1, e_2) de E tels que $I_0(e_1)$ et $I_0(e_2)$ aient un majorant et un minorant commun dans G; il est alors supposé que e_1 et e_2 ont dans E un minorant et un majorant commun (pour la relation $\ll$). En outre, l'auteur introduit (à la suite de M. H. Stone) la notion

$$\operatorname{mid}(e_1, e_2, e_3) = \inf[\sup(e_1, e_2), \sup(e_2, e_3), \sup(e_3, e_1)]$$

(pour des éléments de F), et introduit l'hypothèse que si $f \in F$ est tel que $f \gg$ mid (e_1, e_2, e_3) (e_1 et e_2 étant comme ci-dessus, et $e_3 \in E$) alors il existe $e \in E$ tel que $f \gg e \gg$ mid (e_1, e_2, e_3), et la relation analogue où $\ll$ remplace $\gg$. Un dernier postulat exprime en gros que I_0 (mid (e_1, e_2, e_3)) varie peu lorsque les $I_0(e_i)$ $(1 \leqq i \leqq 3)$ varient peu, mais doit être formulé de façon plus compliquée parce que mid (e_1, e_2, e_3) n'est pas nécessairement dans E.

Cela étant, la marche des idées suit de près la méthode de Daniell: extension de I_0 aux bornes supérieures (resp. inférieures) d'ensembles dénombrables S de E, filtrants pour $\ll$, tels que sup $I_0(S)$ (resp. inf $I_0(S)$) existe dans G (éléments appelés U- et L-éléments); cela permet alors de définir comme d'ordinaire les intégrales supérieure et inférieure de tout élément de F majoré (resp. minoré) par un U-élément (resp. un L-élément); un élément sommable est défini par l'égalité de ces deux intégrales. Les théorèmes de convergence de Lebesgue sont établis sous l'hypothèse supplémentaire que G est "normal" (notion qui a quelque analogie avec la notion de même nom relative aux espaces topologiques). Il faut aussi des hypothèses supplémentaires sur E et F pour étendre la propriété de linéarité de l'intégrale. Les éléments "mesurables" f de F sont définis comme chez Stone, par la condition que pour f', f'' sommables et tels que $f' \leqq f''$, mid (f, f', f'') est sommable; lorsque F est un ensemble de fonctions numériques, on peut aussi naturellement définir la notion d'ensemble mesurable et retrouver la construction de Lebesgue pour l'intégrale.

Le dernier chapitre du livre est consacré aux applications de la théorie; en dehors de la théorie de la mesure, l'auteur montre comment on peut retrouver de cette manière la décomposition spectrale des opérateurs hermitiens et, plus généralement, une théorie spectrale pour certaines algèbres ordonnées. On obtient aussi comme application la généralisation (due à Bochner) du théorème de Bernstein-Widder sur les fonctions complètement monotones; enfin, une dernière application fournit une intégrale apparentée à celle de Perron.

La lecture du livre est malheureusement rendue difficile par l'absence d'un index. *J. Dieudonné* (Evanston, Ill.).

Šmidov, F. I. On the theory of functions of two variables. Doklady Akad. Nauk SSSR (N.S.) **89**, 981–982 (1953). (Russian)

The author defines the notion of a generalized semi-tangent to the graph of a real function $f(x, y)$ of two variables and uses it to strengthen as follows a classical result of Stepanov [Mat. Sbornik **32**, 511–527 (1925)]: For a measurable $f(x, y)$ defined in a plane set E_0, the subset E of E_0 in which f has an approximate differential is a countable sum of sets in each of which f is Lipschitzian and for almost every (x, y) of $E_0 - E$, every ray of vertex $[x, y, f(x, y)]$ is a generalized semi-tangent to the graph of f.

L. C. Young (Madison, Wis.).

De Giorgi, Ennio. Definizione ed espressione analitica del perimetro di un insieme. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **14**, 390–393 (1953).

In Euclidean r-space S, let dG denote the Gaussian distribution, let f be the characteristic function of a Borel set E, and write $f_h(x) = \int f(x+hu)\, dG(u)$, $F_h(B) = \int_B \operatorname{grad} f_h dx$, $P_h(E) = \int |\operatorname{grad} f_h| dx$. The author states without proof the following results: (1) $P(E) = \lim P_h(E)$ as $h \to 0$ exists, and moreover $P(E)$ is finite if and only if it is the total variation of a uniquely determined completely additive $F(B)$ of bounded variation, defined for Borel subsets B of S and with values in S, such that the equation $\int_E \operatorname{grad} g\, dx = \int_S g\, dF$ holds in the clan of real functions $g(x)$ for which g and grad g are continuous and $O(x^{-r-1})$ for large x; furthermore we then have $F(B) = \lim F_h(B)$ wherever $F(B)$ has total variation zero on the boundary of B. (2) $P(E)$ is a semi-continuous function of E which coincides with the minimum limit of its values for approximating polyhedra, and $F(B)$ is weakly continuous in its dependence on the original set E in the class of E for which $P(E)$ is uniformly bounded. (3) The measures of the sets E and $S-E$ cannot both exceed the $r/(r-1)$th power of $P(E)$. *L. C. Young.*

★Caccioppoli, Renato. Elementi di una teoria generale dell'integrazione k-dimensionale in uno spazio n-dimensionale. Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 41–49. Casa Editrice Perrella, Roma, 1953.

The paper is cast in the same mould as a number of others by the same author and with similar titles [cf. these Rev. **13**, 830, 925; **14**, 257, 258]. Lemma I replaces a statement whose erroneous character was pointed out by Radó [Length and area, Amer. Math. Soc. Colloq. Publ., v. 30, New York, 1948, p. 421; these Rev. **9**, 505]; it asserts the existence of an additive extension to all Borel sets for an additive function $\varphi(A)$ of open sets A subject to hypotheses which include $\lim \varphi(A_n) = \varphi(A)$ when A_n definitely contains each closed portion of A. If we choose for A_n repetitions of $A+B$ where B is a fixed open set, this gives $\varphi(A+B) = \varphi(A)$ and by symmetry $\varphi(A+B) = \varphi(B)$, so that $\varphi(A)$ is constant and hence identically zero. This was apparently not intended by the author. *L. C. Young* (Madison, Wis.).

Kudryavcev, L. D. On properties of differentiable mappings of regions of Euclidian spaces. Mat. Sbornik N.S. **32(74)**, 493–514 (1953). (Russian)

The author establishes a number of properties of differentiable mappings and their Jacobians or Jacobian matrices. Those which appear to the reviewer most interesting and novel are as follows, the symbols μ and μ_λ being used to denote Lebesgue measure in Euclidean n-space R_n and Hausdorff measure of dimension λ where $0 \leqq \lambda < \infty$. (I) Let E be a subset of a domain in R_n and f a differentiable mapping of this domain; then the relation $\mu_\lambda(E) = 0$ implies $\mu_\lambda(fE) = 0$ and for a compact E the relation $\mu_\lambda(E) < \infty$ implies $\mu_\lambda(fE) < \infty$; further, if E is measurable and f maps into R_n, the following statement is true for almost every u in fE: let Q describe concentric cubes with diameter tending to zero and centre in $f^{-1}u$ and let $U = fQ$, then $\mu(U - fE)/\mu(U) \to 1$. (II) Let $f(x, y)$ be a differentiable mapping of a domain G of (R_p, R_q) into R_p such that its Jacobian in x, $J(x, y)$, is positive except in a subset G_0 of G, without interior, in which $J(x, y) = 0$; then for each boundary point u of fG the set $f^{-1}u$ is a non-compact subset of G_0. The author stresses incidentally that the equivalence, for a differentiable function of one variable, of possession of a non-negative derivative with the property of being monotone increasing, has as natural generalization, valid for a differentiable mapping into R_n of a domain in R_n, the equivalence of possession of a non-negative Jacobian with the property of being positively oriented. *L. C. Young.*

Cecconi, Jaurés. Sulla identità fra due definizioni di area. Boll. Un. Mat. Ital. (3) **8**, 130–137 (1953).

The Lebesgue area $L(S)$ of a continuous parametric surface S situated in E_3 is shown to agree with its area $A(S)$

in the sense of Reifenberg [Proc. Cambridge Philos. Soc. 47, 687–698 (1951); these Rev. 14, 363]. The author proves this by means of known results in the theory of Lebesgue area and observes that this is in accordance with the remarks of H. Federer, the reviewer of Reifenberg's paper.
L. C. Young (Madison, Wis.).

Theory of Functions of Complex Variables

Watanabe, Yoshikatsu. Ueber die Verschiebung der Nullstellen einiger Funktionen, welche aus Integration gebrochener Ordnung hervorgeht. J. Gakugei Tokushima Univ. 3, 16–20 (1953).

The author gives an elementary proof of the fact [Pólya and Szegö, Aufgaben und Lehrsätze aus der Analysis, Springer, Berlin, 1925, vol. 2, p. 260, problem 179] that the function $H(z)=1+\sum_{n=0}^{\infty}z^n/(\alpha+1)\cdots(\alpha+n)$ has zeros for $\Re(z)>\alpha-1$ or $<\alpha-1$ according as $\alpha>1$ or $1>\alpha>0$. He points out that the functions

$$\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{\Gamma(2n+\alpha+1)},\quad \sum_{n=0}^{\infty}\frac{(-1)^{n-1}z^{2n-1}}{\Gamma(2n+\alpha)},$$

which are related to cos z, sin z as $H(z)$ is related to e^z, have only real zeros for $0\leqq\alpha<2$, $0\leqq\alpha<1$, respectively.
R. P. Boas, Jr. (Evanston, Ill.).

Bertolini, Fernando. Una generalizzazione del teorema di Abel sulle serie di potenze. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 14, 483–488 (1953).

Let $\alpha_0, \alpha_1, \cdots \alpha_n$ be points in the z-plane. Suppose that the points $z_0, z_1, \cdots, z_n$ are such that the circles $|z-\alpha_j|=|z_j-\alpha_j|$ do not intersect.

If the set of numbers $\{a_{0k}(z_0-\alpha_0)^k, a_{jk}(z_j-\alpha_j)^{-k}\}$ $(j=1, 2, \cdots, n;\ k=0, 1, 2, \cdots)$ is bounded, then

$$(1)\qquad \sum_{k=0}^{\infty}a_{0k}(z-\alpha_0)^k+\sum_{k=0}^{\infty}\sum_{j=1}^{n}a_{jk}(z-\alpha_j)^{-k}$$

is absolutely convergent in any closed set inside

$$|z-\alpha_0|<|z_0-\alpha_0|,\quad |z-\alpha_j|>|z_j-\alpha_j|\quad (j=1, \cdots, n).$$

If the series (1) converges for $z=z_j$ $(j=0, 1, \cdots, n)$, then (1) is uniformly convergent in the region of the z-plane defined by

$$0\leqq\frac{|z-z_0|}{|z_0-\alpha_0|-|z-\alpha_0|}\leqq\omega_0,\quad 0\leqq\frac{|z-z_j|}{|z-\alpha_j|-|z_j-\alpha_j|}\leqq\omega_j,$$

where the ω may have any positive value greater than one.
W. H. J. Fuchs (Ithaca, N. Y.).

Tricomi, Francesco G. Determinazione del valore di un classico prodotto infinito. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 14, 3–7 (1953).

The author discusses the product $f(z)=\prod_{n=1}^{\infty}(1-z/a^n)$ in connection with the function $G(z)=\sum_{n=1}^{\infty}(a^n-1)^{-1}z^n/n!$ to which he has recently devoted several papers [same Rend. (8) 11, 141–144 (1951); Comm. Pure Appl. Math. 5, 213–222 (1952); Univ. e Politecnico Torino. Rend. Sem. Mat. 11, 35–46 (1952); these Rev. 13, 637; 14, 259, 632]. If $\gamma(s)$ is the Laplace transform of $G(t)$, $s^{-1}\gamma(s^{-1})=g(s)$, then $g(z)=z\sum_{n=1}^{\infty}(a^n-z)^{-1}$, and $f'(z)/f(z)=-g(z)/z$. The values of $f(z)$ at $z=\pm1$, $\pm a^{\frac{1}{2}}$, and some other points can be calculated in terms of θ-functions, and $f'(a^n)$ can be expressed in terms of $f(1)$.
R. P. Boas, Jr. (Evanston, Ill.).

Jørgensen, Vilhelm. A remark on Bloch's theorem. Mat. Tidsskr. B. 1952, 100–103 (1952).

Let G be the stereographic projection (upon a sphere of radius 1/2 which touches the w-plane at $w=0$) of the image of the disc $|z|<1$ under a meromorphic mapping $w=f(z)$; and at $z=0$, let $|dw/dz|/(1+|w|^2)$, the magnification factor for the mapping from the z-plane upon the sphere, have the positive value a. Then the set G contains a spherical cap with spherical radius arctan a.
G. Piranian.

***Nielsen, Jakob. Some fundamental concepts concerning discontinuous groups of linear substitutions in a complex variable.** Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, pp. 61–70. Johan Grundt Tanums Forlag, Oslo, 1952. 27.50 Kr. (Danish)

Nielsen and Fenchel are writing a book "Topology of surfaces and their transformations" to appear in the Princeton Mathematical Series. In volume I, which is to appear soon, the theory of discontinuous groups of linear fractional transformations will be developed in a new and simplified manner. (This is the theory that is the basis of the Nielsen theory of surface transformations; the more geometric theory of the surface transformations themselves will be contained in volume II.) In the paper under consideration Nielsen gives a brief outline of their new version of this basic theory.
R. H. Fox (Princeton, N. J.).

Simonart, Fernand. Limitations en module d'une fonction linéaire sur un cercle. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 39, 458–462 (1953).

Rudin, Walter. Analyticity, and the maximum modulus principle. Duke Math. J. 20, 449–457 (1953).

If B is the boundary of a compact set K in the plane, we call an algebra A of complex-valued functions continuous on K a maximum modulus algebra on K if for every $f\varepsilon A$ there is a point $z_0\varepsilon B$ such that $|f(z)|\leqq|f(z_0)|$, $z\varepsilon K$. If for each point $s\varepsilon D$ there is an arbitrarily small Jordan domain on which A is a maximum modulus algebra, then we call A a local maximum modulus algebra. The author proves the following theorems. 1) Let A be a maximum modulus algebra on the closure K of a Jordan domain D. If A contains a function ψ which is schlicht (i.e., one-to-one) on K, then every member of A is an analytic function of ψ. If, in addition, A contains a non-constant function φ which is analytic in D, then every member of A is analytic in D. 2) If A is a local maximum modulus algebra in a domain D, and if A contains a function φ which is analytic and not constant in D, then every member of A is analytic in D. 3) Let A be a maximum modulus algebra on the closure K of a domain D which is bounded by a finite number of non-intersecting Jordan curves. If A contains all functions which are analytic and single-valued in D and continuous on K, then every member of A is analytic in D.
H. L. Royden (Stanford, Calif.).

Peschl, Ernst, und Erwe, Friedhelm. Über die Norm regulärer Funktionenspalten. Arch. Math. 4, 191–201 (1953).

Let $f_\nu(z)$ $(\nu=1, 2, \cdots, n)$ be n functions regular in a domain D, (*) $F(z)=\sum_{\nu=1}^{n}|f_\nu(z)|^2$. Any function $F(z)$ defined in G which has a representation (*) is called a norm-function. Let

$$\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right),\quad \frac{\partial}{\partial\bar z}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right),\quad g^{jk}=\frac{\partial^{j+k}g}{\partial z^j\partial\bar z^k}.$$

Theorem A. If $F(z)$ is a norm-function, then

$$|F^{jk}(z)|_{j,k=0,\cdots,r}>0 \ (r=0,1,\cdots,n-1) \qquad (z\in G),$$
$$|F^{jk}(z)|_{j,k=0,\cdots,n}=0 \qquad (z\in G).$$

If G is simply connected, this condition is also sufficient. Theorem B. If $F(z)$ and $G(z)$ are norm-functions, arising from the functions $f_1, \cdots, f_n, g_1, \cdots, g_n$, respectively, and if $F(z)=G(z)$ in a neighborhood of a point $z_0 \in D$, then $(f_1, \cdots, f_n)=(g_1, \cdots, g_n)A'$, where A is a unitary matrix with constant elements. *W. H. J. Fuchs.*

Bernštein, S. N. A necessary and sufficient condition for an almost increasing even function to be a weak weight function. Doklady Akad. Nauk SSSR (N.S.) **90**, 487–490 (1953). (Russian)

The author calls a function $\Phi(x)$ a weak weight function [Izvestiya Akad. Nauk SSSR. Ser. Mat. **16**, 497–502 (1952); these Rev. **14**, 459] if every continuous $f(x)$ vanishing at $\pm\infty$ can be approximated by entire functions of arbitrarily small exponential type with weight $1/\Phi(x)$. By a weight function he means the corresponding thing for weighted polynomial approximation. Here he characterizes weak weight functions which are even, non-decreasing for $x>0$, and satisfy $\Phi(x)\geqq 1$, $x/\Phi(x)\to 0$. His necessary and sufficient condition is that, as $H_p(x)$ runs through the class of entire functions of fixed exponential type $p>0$ with even modulus, such that $H_p(0)\neq 0$ and $|H_p(x)|\leqq\Phi(x)$,

$$\sup |H_p(0)| \sum_{k=1}^{\infty} \frac{2|\beta_k|}{\alpha_k^2+\beta_k^2}=\infty,$$

where $\alpha_k+i\beta_k$ are the zeros of $H_p(x)$. Corollary: $\Phi(x)$ (with the properties stated above) is a weak weight function if and only if it is an antimajorant [as defined in the paper cited above]. If $\Phi(x)$ is not a weight function and $f(x)$ admits polynomial approximation with weight $1/\Phi(x)$, then $f(x)$ has the special form $\exp(hx^2)S(x)$, where $S(x)$ is an entire function of order 1 whose zeros $z_n\neq 0$ satisfy $\sum 1/|z_n|<\infty$, $\sum|\Im(1/z_n)|<\infty$. Similarly for functions which are not weak weight functions. *R. P. Boas, Jr.*

Bernštein, S. N. Weak weight functions and majorants. Doklady Akad. Nauk SSSR (N.S.) **90**, 703–706 (1953). (Russian)

The author continues the paper reviewed above. In the terminology of his earlier papers [references in the preceding review], he shows that an even positive function $\Phi(x)$, increasing for $x>0$, is a weak weight function if and only if the supremum of $\int_{-\infty}^{\infty} x^{-2}\log|G_p(x)/G_p(0)|dx$ is infinite, where the supremum is taken over all entire $G_p(x)$ of given exponential type p and even modulus $|G_p(x)|\leqq\Phi(x)$. A function $\Phi(x)$ as above is either a weak weight function or a majorant of quasi-finite growth; it is also either a weight function or a majorant of quasi-finite growth with respect to polynomials. Several other corollaries are given. In particular, if $F(x)$ is measurable and $F(x)>c>0$, the condition $\int_{-\infty}^{\infty}(1+x^2)^{-1}\log F(x)dx<\infty$ is sufficient for $F(x)$ to be a majorant of finite growth, and hence necessary for $F(x)$ to be a weak weight function or a weight function.

R. P. Boas, Jr. (Evanston, Ill.).

Levin, B. Ya. Operators preserving inequalities between entire functions. Doklady Akad. Nauk SSSR (N.S.) **89**, 605–608 (1953). (Russian)

In an earlier paper [same Doklady **79**, 397–400 (1951); these Rev. **13**, 122] the author characterized $\mathfrak{B}$-operators, which have a property generalizing S. Bernstein's inequality for the derivative of an entire function of exponential type. Here he characterizes similarly the $\mathfrak{B}^*$-operators; these are defined on the linear manifold spanned by P^* (the class of entire functions of the form $\omega(z)=e^{-\gamma z^2}\omega_1(z)$, $\gamma\geqq 0$, $\omega_1(z)$ of order 1, with no zeros in the lower half plane, and such that $|\omega_1(z)|\leqq|\omega_1(\bar{z})|$ in the lower half plane), and leave P^* invariant. The generalized Bernstein inequality is thus extended to a class of functions more general than those of exponential type. *R. P. Boas, Jr.* (Evanston, Ill.).

Wilson, R. The coefficient theory of integral functions with dominant exponential parts. Quart. J. Math., Oxford Ser. (2) **4**, 142–149 (1953).

The theory applies to entire functions of exponential type whose Laplace transforms are meromorphic on the circle of convergence. The entire functions then have dominant exponential parts arising from the poles on the circle of convergence, secondary parts arising from poles on a smaller circle, and so on. The author applies the Hadamard coefficient theory to obtain necessary and sufficient conditions for this to happen, and to obtain expressions for the exponents which occur. Extensions are given to functions of general finite order. [In formulas (4), (13), (14), divide $A_{\mu\nu}$ by $(\mu-1)!$; in Theorem 3, replace the exponent $1/n$ by ρ/n.] *R. P. Boas, Jr.* (Evanston, Ill.).

Singh, Udita Narayana. Fonctions entières et transformée de Fourier généralisée. C. R. Acad. Sci. Paris **237**, 14–16 (1953).

A measurable function f, defined on the real axis, is said to be of class $C(k)$ $(0\leqq k<\infty)$ if $\int_0^x|f(t)|dt=O(|x|^k)$ as $|x|\to\infty$. The following theorem is announced. Let f be an entire function of exponential type and of class $C(k)$, and suppose

$$(*)\quad \limsup_{r\to\infty} r^{-1}\log|f(-ir)|\leqq h, \quad \limsup_{r\to\infty} r^{-1}\log|f(ir)|\leqq h'.$$

Then the generalized Fourier transform of f (in the sense of Carleman) vanishes uniformly on every compact interval outside $[-h', h]$. Conversely, if the conclusion holds for some $F\in C(k)$, then F coincides almost everywhere on the real axis with an entire function f of exponential type for which (*) holds. A second theorem concerns pairs of functions, analytic in the upper and lower half planes, and their generalized Fourier transforms. *W. Rudin.*

Hayman, W. K. The minimum modulus of large integral functions. Proc. London Math. Soc. (3) **2**, 469–512. (1952).

This paper contains the awaited details of the author's solution of Wiman's problem dealing with the behavior of $\sigma(r,f)=\log m(r)/\log M(r)$ where $m(r)$ and $M(r)$ are the minimum and maximum of $|f(z)|$ on $|z|=r$, and f is an entire function. Littlewood (1908) showed that there is a constant $C(k)$ such that $\limsup\sigma(r,f)\geqq -C(k)$ for all f of order k. Valiron and Wiman (1914) proved that for $0<k<1$, $C(k)=-\cos\pi k$, and Wiman (1918) proved that for any entire function of the form $f(z)=e^{g(z)}$, $\limsup\sigma(r,f)\geqq -1$, leading to the conjecture that this holds also for unrestricted f. The author recently announced results which show this to be false [C. R. Acad. Sci. Paris **232**, 591–593 (1951); these Rev. **12**, 689]. For functions of finite order, the author now shows that the correct order of magnitude of $C(k)$ is $\log k$; more precisely, if f has order $k>k_0$ then $\limsup\sigma(r,f)\geqq -A\log k$ where the limit can in fact be

taken in a set of r of positive lower logarithmic density, and where A depends upon k_0, but (for large k_0) obeys $0<A<2.2$. For functions of arbitrary growth, the author obtains the general relation $\sigma(r, f) \geqq -A \log\log\log M(r)$ holding for a set of r of positive lower logarithmic density (positive lower density if f has non zero lower order) where the constant A obeys $.09<A<2.2$. In another direction, the author proves that there is a constant A_0 such that if $|f(z)|M(r)^{A_0} \leqq 1$ on an unbounded connected set Γ, then f is a constant. Beurling [Duke Math. J. **16**, 355–359 (1949); these Rev. **10**, 692] proved a similar theorem in which the set is specialized to a ray, and A_0 may be taken as any number larger than 1. The proofs are based upon a detailed estimation of a general subharmonic function of circles $|z|=r$, based largely upon the Poisson-Jensen formula. The second half of the paper is devoted to the construction of counterexamples to obtain the lower bounds on the constants A. [See also Boas, Buck, and Erdös, Amer. J. Math. **70**, 400–402 (1948); these Rev. **9**, 577.] *R. C. Buck.*

Bose, S. K. Maximum and minimum function of a meromorphic function. Bull. Calcutta Math. Soc. **44**, 69–74 (1952).

Bose, S. K. Some properties of the maximum function of a meromorphic function. Math. Z. **56**, 223–226 (1952).

In the first paper the author considers a meromorphic function $f(z)=f_1(z)/P(z)$, where $f_1(z)$ is entire and $f_2(z)$ is a canonical product. He defines a "maximum function" as $S(r)=M_1(r)/m_2(r)$, where $M_1(r)$ is the maximum modulus of $f_1(z)$ and $m_2(r)$ is the minimum modulus of $P(z)$ outside a set of circles of finite total length surrounding the poles. Let $S^*(r)$ be the maximum function of $f(z)$ if the order of $f_1(z)$ is at least as large as that of $P(z)$, and otherwise let $S^*(r)$ be the maximum function of $1/f(z)$. Then the order of $f(z)$ is defined as lim sup log log $S^*(r)/\log r$. [This is not precisely the author's statement, but seems to be what he intends.] The author shows that this order is the same as that defined by Nevanlinna and by Shimizu. He also advances a proof that log $S(r)$ is a convex function of log r, but the reviewer does not find it convincing. The author gives applications to an interpolation problem for entire functions and to a generalization of Schwarz's lemma.

The second paper is devoted to theorems such as the following: $\limsup r^{-\rho} \log S(r) = \limsup r^{-\rho} \log M_1(r)$ if $f(z)$ and $f_1(z)$ are both of order ρ. The author's proofs seem to be defective since they disregard the possibility that log $m_2(r)$ may be larger than the order of $P(z)$ would suggest [for examples of this possibility see the paper reviewed above]. S. M. Shah and S. K. Singh inform the reviewer that they have constructed explicit counterexamples to the first three theorems of the author's second paper. *R. P. Boas, Jr.*

Mikhail, M. N. Basic sets of polynomials and their reciprocal, product and quotient sets. Duke Math. J. **20**, 459–479 (1953).

Mikhail, M. N. On the order of the reciprocal set of a basic set of polynomials. Pacific J. Math. **3**, 617–623 (1953).

The author proves many results about basic sets of polynomials [terminology as in J. M. Whittaker, Sur les séries de base des polynomes quelconques, Gauthier-Villars, Paris, 1949; these Rev. **11**, 344]. First he gives a new formula for calculating the order of a wide class of basic sets. This enables him to prove some inequalities for the order of reciprocal sets and product sets, generalizing earlier results of several authors. Next he gives some inequalities for the range of effectiveness of reciprocal, product and quotient sets. Then basic sets with bounded coefficients are discussed. The second paper contains another inequality for the order of a reciprocal set. The author's results cannot be described in detail in a short review. *R. P. Boas, Jr.*

Iliev, Lyubomir. Series of Faber polynomials whose coefficients assume a finite number of values. Doklady Akad. Nauk SSSR (N.S.) **90**, 499–502 (1953). (Russian)

If $f(z)$ is represented by (1) $\sum c_n \Phi_n(z)$, a series of Faber polynomials whose coefficients c_n assume only a finite number of different values, then either the sequence $\{c_n\}$ is ultimately periodic, or else the boundary of the region of convergence of (1) is a natural boundary for the function $f(z)$. The author's proof follows closely Szegö's proof of the theorem for the special case where (1) is a Taylor series [S.-B. Preuss. Akad. Wiss. **1922**, 88–91]. *G. Piranian.*

Charzyński, Zygmunt. Sur les fonctions univalentes bornées. Rozprawy Mat. **2**, 58 pp. (1953).

The author devotes most of this article to a derivation of a differential equation satisfied by an extremal bounded univalent function. Let F_T be the family of analytic functions $f(z)=\sum_{n=1}^\infty a_n z^n$, $a_1 \geqq T>0$, which are univalent and bounded ($|f(z)|<1$) in the unit circle $|z|<1$. Let

$$a_n/a_1 = X_n + iY_n, \quad n=2, 3, 4, \cdots,$$

and

$$H_f = H(X_2, \cdots, X_N, Y_2, \cdots, Y_N), \quad N \geqq 2,$$

be any real function of the $2N-2$ real variables, with continuous first partial derivatives. Then there exist extremal functions $f^*(z)$ in F_T for which H_f assumes its largest value. Each extremal satisfies the differential equation

$$[f^{*\prime}(z)/f^*(z)]^2 \mathfrak{M}[f^*(z)] = z^{-2}\mathfrak{N}(z), \quad 0 \leqq |z| < 1,$$

where

$$\mathfrak{M}(w) = \left[\sum_{p=2}^N \frac{\mathfrak{D}^*_{p-1}}{w^{p-1}} + \overline{\mathfrak{D}^*}_{p-1} w^{p-1}\right] - 2\mathfrak{P}^*,$$

$$\mathfrak{N}(z) = \left[\sum_{p=2}^N \frac{\mathfrak{E}^*_{p-1}}{z^{p-1}} + \overline{\mathfrak{E}^*}_{p-1} z^{p-1}\right] - 2\mathfrak{P}^*,$$

$$[f^*(z)]^p = \sum_{n=1}^\infty a_n^{*(p)} z^n, \quad p=2, 3, \cdots,$$

$$f^*(z) = \sum_{n=1}^\infty a_n^* z^n, \quad a_n^*/a_1^* = X_n^* + iY_n^*, \quad n=2, 3, \cdots,$$

$$H_n^* = H_{X_n}(X_2^*, \cdots, X_N^*, Y_2^*, \cdots, Y_N^*) - iH_{Y_n}(X_2^*, \cdots, X_N^*, Y_2^*, \cdots, Y_N^*), \qquad n=2, \cdots, N,$$

$$\mathfrak{D}^*_{p-1} = 2\sum_{n=p}^N a_n^{*(p)} H_n^*, \quad p=2, \cdots, N,$$

$$\mathfrak{E}_0^* = \sum_{n=2}^N (n-1) a_n^* H_n^*, \quad \mathfrak{E}^*_{p-1} = 2\sum_{n=p}^N (n-p+1) a_n^* H_n^*, \qquad p=2, \cdots, N,$$

and

$$\mathfrak{P}^* = \min_{0 \leqq \varphi \leqq 2\pi} R\left\{\sum_{p=2}^N \mathfrak{D}^*_{p-1} e^{i\varphi(p-1)}\right\}.$$

$\mathfrak{M}(w)$ and $\mathfrak{N}(z)$ take on only real, non-negative values on

the circumferences $|w|=1$ and $|z|=1$ resp., and each has at least one double zero there. Also, $a_1^*=T$. A more general result had previously been published by the author with W. Janowski [Ann. Univ. Mariae Curie-Skłodowska. Sect. A. 4, 41–56 (1950); these Rev. 13, 122]. The last few pages are devoted to proving the following orthogonality relation for the coefficients $a_n^{(p)}$ in $f(z)^p=\sum_{n=1}^{\infty}a_n^{(p)}z^n$ where $f(z)$ is a bounded univalent function in $|z|<1$ which transforms $|z|<1$ into a region with the same measure:

$$\sum_{n=1}^{\infty}\left(\frac{n}{k}\right)^{1/2}a_n^{(k)}\overline{\left(\frac{n}{m}\right)^{1/2}a_n^{(m)}}=\begin{cases}0 \text{ for } k\neq m\\ 1 \text{ for } k=m,\end{cases}\quad k, m=1, 2, \cdots.$$

G. Springer (Evanston, Ill.).

Ladegast, Konrad. Beiträge zur Theorie der schlichten Funktionen. Math. Z. 58, 115–159 (1953).

Let $F(z)$ be a schlicht function in $|z|<R$ with its only pole at $z=p$ $(0<|p|<R)$ and with principal part $P/(z-p)$ and power series expansion $F(z)=a_0+a_1z+a_2z^2+\cdots$ in $|z|<p$. Using a generalized area theorem, the author proves that in $|z|<R$,

$$\left|\frac{F'(z_1)F'(z_2)}{[F(z_2)-F(z_1)]^2}-\frac{1}{(z_2-z_1)^2}\right|\leq\frac{R^2}{(R^2-|z_1|^2)(R^2-|z_2|^2)}.$$

This inequality is precise in the sense that equality is assumed for certain functions given by the author. Furthermore, the inequality is invariant under linear fractional transformations of the independent variable which take $|z|<R$ into itself, and under arbitrary linear fractional transformations of the dependent variable. From this the author derives numerous other distortion theorems for schlicht functions in $|z|<R$, which (a) are regular in $|z|<R$, (b) have a simple pole at $z=p$ as does $F(z)$ above, or (c) have a single simple pole at $z=0$. He also gets estimates for the radius of schlichtness in terms of the coefficients in the series expansion of the function. *G. Springer.*

Zin, Giovanni. Sull'esistenza in un dominio jordaniano di funzioni olomorfe all'interno e convergenti al contorno verso valori assegnati. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 14, 476–483 (1953).

Let D be a region in the z-plane bounded by a rectifiable Jordan curve C with parametrization $z(s)=x(s)+iy(s)$ $(0\leq s\leq L)$ in terms of the arclength s. Let $\varphi(z)$ be a given function of bounded variation on C (i.e., $|\varphi(z(s))|$ is of bounded variation in $0\leq s\leq L$) such that $\int_C\varphi(z)z^ndz=0$ $(n=0, 1, 2, \cdots)$. It is shown (A) that $\varphi(z^+)=\varphi(z^-)$ for each point z on C; (B) that the function $f(z)$ defined in $D+C$ by $f(z)=(2\pi i)^{-1}\int_C\varphi(t)(t-z)^{-1}dt$ for $z\in D$, $f(z)=\varphi(z^+)=\varphi(z^-)$ for $z\in C$, is continuous in $D+C$; and (C) that every function $\Phi(z)$ which is analytic in D and continuous in $D+C$ has the property that $\int_0^L\varphi(z(s))[d\Phi(z(s))/ds]ds=0$ whenever $|\Phi(z(s))|$ is absolutely continuous in $0\leq s\leq L$. This theorem extends an earlier result of the author [Ann. Mat. Pura Appl. (4) 34, 365–405 (1953); these Rev. 14, 1073].

A. J. Lohwater (Ann Arbor, Mich.)

✱Nehari, Zeev. Dirichlet's principle and some inequalities in the theory of conformal mapping. Contributions to the theory of Riemann surfaces, pp. 167–175. Annals of Mathematics Studies, no. 30. Princeton University Press, Princeton, N. J., 1953. $4.00.

Soient D et D^* $(D\subset D^*)$ deux domaines portés par une même surface de Riemann de genre fini; soit $S(z)$ harmonique et uniforme dans D, excepté en un nombre fini de points. S'il existe une fonction p définie dans D, nulle sur C, et telle que $h=p+S$ soit harmonique et uniforme dans D, et des fonctions analogues p^*, h^* relatives à D^*, alors on a:

$$(1)\qquad \int_C h\frac{\partial p}{\partial n}ds\geq\int_{C^*}h^*\frac{\partial p^*}{\partial n}ds,$$

les dérivées étant prises sur la normale extérieure. De cette inégalité, qu'il démontre très simplement, l'auteur déduit nombre d'applications intéressantes, dont certaines avaient été déjà établies par d'autres méthodes: inégalités concernant les fonctions de Green, les noyaux de Bergmann-Schiffer et la représentation conforme. Généralisations de (1) concernant: (a) les domaines multiplement connexes, $p=0$ étant remplacé par $p=$Cte sur les frontières non intérieures; (b) le cas de domaines disjoints. Application de la méthode à la résolution d'un problème extrémal.

J. Lelong (Lille).

✱Schaeffer, A. C. An extremal boundary value problem. Contributions to the theory of Riemann surfaces, pp. 41–47. Annals of Mathematics Studies, no. 30. Princeton University Press, Princeton, N. J., 1953. $4.00.

Let E be a closed set of positive measure on $|z|=1$. It is proved that E has a perfect subset E_0 with the same measure as E such that there is a function $f(z)=a_1z+\cdots$ regular and schlicht in $|z|<1$ mapping $|z|<1$ onto $|w|<1$ cut by radial slits, the set E_0 mapping onto $|w|=1$ and the complement of E_0 mapping onto the radial slits. This result is a consequence of the following extremal theorem. For the set of functions $f(z)=a_1z+\cdots$ regular and schlicht in $|z|<1$ with $\lim_{r\to1^-}|f(re^{i\theta})|\geq1$ for almost all $e^{i\theta}\in E$, $|a_1|$ attains a minimum for a function which maps E_0 onto $|w|=1$ and its complement onto radial slits. *A. W. Goodman.*

Huckemann, Friedrich. Verschmelzung von Randstellen Riemannscher Flächen. Mitt. Math. Sem. Univ. Giessen no. 41, i+36 pp. (1952).

The Riemann surface which belongs to the function $\sin z$ has infinitely many branch-points over ±1. If the surface is deformed by letting the branch-points take positions $p_\nu\to+\infty$ and $n_\nu\to+\infty$ it may happen that the order of the corresponding entire function is decreased. This phenomenon is studied in quite some detail by geometric methods. There are no references to similar work by F. Ulrich [Duke Math. J. 5, 567–589 (1939); these Rev. 1, 8] and G. R. MacLane [Trans. Amer. Math. Soc. 62, 99–113 (1947); these Rev. 9, 85]. *L. Ahlfors.*

✱Kaplan, Wilfred. Construction of parabolic Riemann surfaces by the general reflection principle. Contributions to the theory of Riemann surfaces, pp. 103–106. Annals of Mathematics Studies, no. 30. Princeton University Press, Princeton, N. J., 1953. $4.00.

The following principle is used to construct various Riemann surfaces of prescribed type: Let $w=f(z)$ be analytic in a simply-connected domain D, let $v=\text{Im}[f(z)]$ have constant boundary values on an open analytic curve C which forms part of the boundary of D, and let there exist a one-to-one conformal mapping $w=\phi(z)$ of D on the upper half-plane which transforms C onto the entire real axis, then $f(z)$ can be continued analytically wherever $\phi(z)$ can be continued. In particular, if $\phi(z)$ is an entire function, then $f(z)$ is entire. *H. L. Royden* (Stanford, Calif.).

***Schiffer, Menahem. Variational methods in the theory of Riemann surfaces.** Contributions to the theory of Riemann surfaces, pp. 15–30. Annals of Mathematics Studies, no. 30. Princeton University Press, Princeton, N. J., 1953. $4.00.

The early attempts at varying a Riemann surface are reviewed, and they are linked with the modern variational theory through introduction of the double of a surface with boundary. Several new variations of Riemann surfaces are defined, some of which alter the genus, and applications are made to a variety of extremal problems. Mention is made of surface imbedded in space. *P. R. Garabedian.*

***Heins, Maurice. A problem concerning the continuation of Riemann surfaces.** Contributions to the theory of Riemann surfaces, pp. 55–62. Annals of Mathematics Studies, no. 30. Princeton University Press, Princeton, N. J., 1953. $4.00.

The following theorem is established. A Riemann surface F admits continuations to all compact Riemann surfaces G of a given genus $p\geqq 1$ if and only if F is conformally equivalent to a bounded plane region. The author first shows that, if F can be continued to all G with $p>1$, then it can be continued to all G with $p=1$. The antithesis that F is not planar then leads to the contradiction that the set of the principal moduli of the G with $p=1$ is bounded.

L. Sario (Cambridge, Mass.).

***Royden, H. L. On the ideal boundary of a Riemann surface.** Contributions to the theory of Riemann surfaces, pp. 107–109. Annals of Mathematics Studies, no. 30. Princeton University Press, Princeton, N. J., 1953. $4.00.

The author discusses the possibility of presenting the ideal boundary of a Riemann surface by a subset of the maximal ideals in the Gelfand space associated with the algebra of functions which are bounded and have a finite Dirichlet integral. *L. Sario* (Cambridge, Mass.).

***Fourès, Léonce. Coverings of Riemann surfaces.** Contributions to the theory of Riemann surfaces, pp. 141–155. Annals of Mathematics Studies, no. 30. Princeton University Press, Princeton, N. J., 1953. $4.00.

English translation of Ann. Sci. Ecole Norm. Sup. (3) **69**, 183–201 (1952); these Rev. **14**, 550. *L. Sario.*

***Kakutani, Shizuo. Random walk and the type problem of Riemann surfaces.** Contributions to the theory of Riemann surfaces, pp. 95–101. Annals of Mathematics Studies, no. 30. Princeton University Press, Princeton, N. J., 1953. $4.00.

A Riemann surface Σ^n over real, n-dimensional, Euclidean space R^n is a connected topological space on which a continuous map φ: $\Sigma^n\rightarrow R^n$ is defined such that any $\zeta\in\Sigma^n$ has a neighborhood which is mapped homeomorphically by φ on an n-dimensional sphere with center $\varphi(\zeta)$. A Brownian motion on Σ^n is defined in the obvious way as the φ^{-1}-image of a Brownian motion in R^n. Let F be a closed set in Σ^n, $u(\zeta_0, F)$ the probability that a Brownian motion starting from ζ_0 will reach F. The author states that 2 cases (I, II) can arise. (I) Almost all Brownian paths starting at ζ_0 are dense on Σ^n and revert to an assigned open set after arbitrarily long intervals of time. Then, for any $\zeta_0\in\Sigma^n-F$, $u(\zeta_0, F)=0$ if F has zero $(n-2)$-dimensional capacity [logarithmic capacity for $n=2$], $u(\zeta_0, F)=1$, if F has positive $(n-2)$-dimensional capacity. (II) Almost all Brownian motions tend to the boundary of Σ^n (i.e., given a compact $E\subset\Sigma^n$, the Brownian motion ultimately is completely outside E). $u(\zeta_0, F)=0$ if and only if $\zeta_0\in\Sigma^n-F$ and F is of zero $(n-2)$-dimensional capacity. $u(\zeta_0, F)<1$, if Σ^n-F has one unbounded component C and $\zeta_0\in C$. As $\zeta_0\in C$ approaches the boundary of Σ^n, $u(\zeta_0, F)\rightarrow 0$.

W. H. J. Fuchs (Ithaca, N. Y.).

Slepenčuk, K. M. Representation of an analytic function of two variables by means of a double infinite product. Uspehi Matem. Nauk (N.S.) **8**, no. 2(54), 139–142 (1953). (Russian)

J. F. Ritt [Math. Z. **32**, 1–3 (1930)] proved that an analytic function $f(x)$ with $f(0)=1$ has a unique representation as an infinite product $\prod_{m=1}^{\infty}(1+a_mx^m)$. The author states an analogous theorem concerning representation of an analytic function $f(x, y)$ as a product $\prod_{m,n=0}^{\infty}(1+a_{mn}x^my^n)$. His proof, however, is not valid because the recursion formula by means of which he calculates a_{mn}, is wrong.

H. Tornehave (Copenhagen).

Hitotumatu, Sin. Two remarks on my paper "A note on the maximal ideals of analytic functions". Kōdai Math. Sem. Rep. **1953**, 31 (1953).

See same Rep. **1952**, 51–53; these Rev. **14**, 264.

***Strehlke, Karl. Planarkonvexe Bereiche im Raum von n komplexen Veränderlichen.** Dissertationen der mathematisch-naturwissenschaftlichen Fakultät der westfälischen Landesuniversität Münster in Referaten, Heft 1, pp. 15–17. Aschendorffsche Verlagsbuchhandlung, Münster, 1952.

***Bremermann, Hans-Joachim. Die Charakterisierung von Regularitätsgebieten durch pseudokonvexe Funktionen.** Dissertationen der mathematisch-naturwissenschaftlichen Fakultät der westfälischen Wilhelms-Universität zu Münster in Referaten, Heft 2, pp. 5–7. Aschendorffsche Verlagsbuchhandlung, Münster, 1952.

***Will, Herbert. Approximation regulärer Funktionen mehrerer Veränderlichen in komplexen Mannigfaltigkeiten.** Dissertationen der mathematisch-naturwissenschaftlichen Fakultät der westfälischen Wilhelms-Universität zu Münster in Referaten, Heft 3, pp. 5–7. Aschendorffsche Verlagsbuchhandlung, Münster, 1953.

***Koch, Karl. Zur Theorie der Funktionen mehrerer komplexer Veränderlichen. Die analytische Projektion.** Dissertationen der mathematisch-naturwissenschaftlichen Fakultät der westfälischen Wilhelms-Universität zu Münster in Referaten, Heft 3, pp. 8–9. Aschendorffsche Verlagsbuchhandlung, Münster, 1953.

***Conforto, Fabio. Un nuovo indirizzo nella teoria delle funzioni abeliane.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. I, pp. 141–163. Casa Editrice Perrella, Roma, 1953.

This paper contains a rather complete exposition of the definitions, results, and methods pertaining to the problems of the theory of abelian functions and modular functions of several variables, which are expounded in detail in the author's book "Funzioni abeliane modulari, vol. 1" [Edizioni Univ. "Docet", Roma, 1952; these Rev. **14**, 859].

O. F. G. Schilling (Chicago, Ill.).

Theory of Series

Szász, O. On the product of two summability methods. Ann. Soc. Polon. Math. **25** (1952), 75–84 (1953).

Let T_1 and T_2 be regular transformations for evaluation of sequences. It is an open question under what conditions $T_1 \cdot T_2 \supset T_1$, that is, the T_1 transform of the T_2 transform of a sequence s_n converges to s whenever the T_1 transform converges to s. The author [Proc. Amer. Math. Soc. **3**, 257–263 (1952); these Rev. **13**, 835] has given several specific pairs of transformations T_1 and T_2 for which $T_1 \cdot T_2 \supset T_1$, and the present paper gives more pairs. The relation $T_1 \cdot T_2 \supset T_1$ is valid when (i) $T_1 = A$, the Abel method and $T_2 = H$, a regular Hausdorff method; when (ii) $T_1 = B$, the Borel exponential method and $T_2 = H$; and when (iii) $T_1 = A$ and T_2 is a regular Hardy-Littlewood circle method. An example is given to show that the relation $T_1 T_2 \supset T_1$ fails to hold when $T_1 = B$ and T_2 is the binary transformation for which $t_n = \frac{1}{2}(s_n + s_{n+1})$. (A simpler example is that for which T_1 and T_2 are the transformations $t_n^{(1)} = s_{2n}$ and $t_n^{(2)} = s_{2n+1}$. Here the T_1 transform of $s_0, s_1, s_2, \cdots$ is $s_0, s_2, s_4, s_6, \cdots$ and the $T_1 \cdot T_2$ transform is $s_1, s_5, s_9, s_{13}, \cdots$. Failure of the relation $T_1 \cdot T_2 \supset T_1$ follows from existence of sequences s_n for which $\lim s_{2n}$ exists and $\lim s_{4n+1}$ fails to exist.)

R. P. Agnew (Ithaca, N. Y.).

Wollan, G. N. On Euler methods of summability for double series. Proc. Amer. Math. Soc. **4**, 583–587 (1953).

Beim Problem der Indexverschiebung beim Euler-Verfahren spielen die Matrizen (E_q) und (E_q') mit den Elementen

$$(q+1)^{-n}\binom{n}{k}q^{n-k} \text{ bzw. } (q+1)^{-n-1}\binom{n+1}{k+1}q^{n-k}$$

$$(0 \leqq k \leqq n;\ n=0, 1, \cdots;\ q \geqq 0)$$

eine Rolle. (E_q) und (E_q') sind äquivalent [G. H. Hardy, Divergent series, Oxford, 1949, S. 180; diese Rev. **11**, 25]. Verf. betrachtet die für Doppelfolgen entsprechenden Matrizen $(\bar{E}_q)$ und $(\bar{E}_q')$ mit den Elementen

$$(q+1)^{-m-n}\binom{m}{h}\binom{n}{k}q^{m+n-h-k}$$

bzw.

$$(q+1)^{-m-n-2}\binom{m+1}{h+1}\binom{n+1}{k+1}q^{m+n-h-k}$$

$$(0 \leqq h,\ k \leqq m, n;\ m, n=0, 1, \cdots;\ q \geqq 0)$$

und zeigt, dass aus $\bar{E}_q'-\lim s_{mn} = s$ stets $\bar{E}_q-\lim s_{mn} = s$ folgt; die Umkehrung ist im allgemeinen falsch, bei beschränkter $\bar{E}_q$-Transformation stets richtig. Weiter wird ein o-Umkehrsatz für das $\bar{E}_1$-Verfahren bewiesen: Aus $\bar{E}_1-\lim s_{mn} = s$, $|s_{mn}| < A$ $(m, n \geqq 0)$ und $\lim (m^{1/2}+n^{1/2})(mn)^{1/2}a_{mn} = 0$ folgt $\lim s_{mn} = s$ $(s_{mn} = \sum_{h,k=0}^{m,n} a_{hk}$ gesetzt). *D. Gaier.*

Agnew, R. P. Tauberian series and their Abel power series transforms. Ann. Soc. Polon. Math. **25** (1952), 218–230 (1953).

Soit $\sum u_n$ une série à termes réels ou complexes, telle que $\limsup |nu_n| < \infty$. Soit $S(x) = \sum_{0 \leq k \leq x} u_k$, a un nombre >0, C la constante d'Euler. On a:

$$(1) \quad \limsup_{t \to 1-0} \left| \sum_0^\infty t^k u_k - S\left(\frac{9}{1-t}\right) \right| \leqq A(a) \cdot \limsup |nu_n|$$

où $A(a) = C + \log a + 2\int_a^\infty e^{-x} x^{-1} dx$. Il existe une série à termes réels pour laquelle l'égalité a lieu dans (1). Le résultat est encore vrai si $t \to 1$ par les valeurs $t_n = 1 - a/n$ $(n \to \infty)$. Diverses généralisations et compléments où varient en particulier les hypothèses faites sur lim sup et lim inf de $n(1-t_n)$. *M. Zamansky* (Paris).

Chow, H. C. On the summability of a power series. Quart. J. Math., Oxford Ser. (2) **4**, 152–160 (1953).

Die Potenzreihe (1) $f(z) = \sum a_n z^n$ mit $\limsup |a_n|^{1/n} = 1$ heisst für $z=1$ $|C_k|$-summierbar $(k>0)$, wenn

$$\sum |\sigma_n - \sigma_{n-1}| < \infty$$

ist, wobei σ_n die C_k-Mittel der Reihe $\sum a_n$ sind. Eine notwendige Bedingung dafür ist (2) $\sum n^{-k}|a_n| < \infty$. Verf. gibt verschiedene Bedingungen über das Verhalten von $f(z)$ bei $z=1$ an, die zusammen mit (2) die $|C_k|$-Summierbarkeit von $\sum a_n$ zur Folge haben. Satz 1: Sei $(\alpha\beta)$ ein $e^{i\theta}$ enthaltender Bogen von $|z|=1$ und (2) erfüllt für $k>0$. Dann ist notwendig und hinreichend für die $|C_k|$-Summierbarkeit von $\sum a_n e^{in\theta}$, dass

$$\sum_{n=1}^\infty n^{-1-k} \left| \int_{re^{i\alpha}}^{re^{i\beta}} f'(z)\{(e^{i\theta}-z)^{-k} + c_0 + c_1 z\} z^{-n} dz \right| < \infty$$

ist, wobei $r = 1 - 1/n$ ist und $c_0(\theta)$ und $c_1(\theta)$ von n nicht abhängen. Satz 3: Ist $f'(z) = O(|1-z|^{-\gamma})$ in dem in $|z|<1$ gelegenen Teil einer Umgebung von $z=1$ $(0 \leqq \gamma < 1)$, und ist (2) erfüllt für $k>0$, so existiert $|C_k| - \sum a_n$. Satz 3 wird falsch für $\gamma = 1$. Weiter wird $|C_k|$-Summierbarkeit fast überall auf $(\alpha\beta)$ untersucht. Ein Sonderfall von Satz 3 sei hervorgehoben: Ist (1) in $z=1$ regulär, so ist $\sum a_n$ genau dann $|C_k|$-summierbar, wenn (2) gilt. Beweistechnisch schliesst sich die Arbeit an verwandte Untersuchungen von M. Riesz [J. Reine Angew. Math. **140**, 89–99 (1911)] und Offord [Proc. London Math. Soc. (2) **33**, 467–480 (1932)] an.

D. Gaier (Cambridge, Mass.).

Gumowski, Igor. Summation of slowly converging series. J. Appl. Phys. **24**, 1068; erratum, 1330 (1953).

Without indicating that his results are well known, the author uses formal operational methods to obtain the formula obtained by setting $N = \infty$, $f(N) = 0$, and $f^{(2k-1)}(N) = 0$ in the Euler (or Euler-Maclaurin) summation formula

$$\sum_{k=n}^N f(k) = \int_n^N f(x)dx + \frac{f(n)+f(N)}{2} + \sum_{k=1}^\infty \frac{B_{2k}}{(2k)!}[f^{(2k-1)}(N) - f^{(2k-1)}(n)].$$

R. P. Agnew (Ithaca, N. Y.).

Smith, R. C. T. Inverse factorial series. Quart. J. Math., Oxford Ser. (2) **4**, 132–135 (1953).

The author proves that if the coefficients in the series

$$\Gamma(z)A(z) = \sum_{n=0}^\infty \frac{n! a_n}{z(z+1)\cdots(z+n)}$$

satisfy $\sum 2^n |a_n| < \infty$ then the series converges for all finite z and is represented by the one-sided interpolation series

$$\Gamma(z)A(z) = \sum_{r=0}^\infty \frac{(-1)^r A(-r)}{r!(z+r)}.$$

R. P. Boas, Jr. (Evanston, Ill.).

Popken, J. Asymptotic expansions from an algebraic standpoint. Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15**, 131–143 (1953).

Let G be an unbounded subset of the complex plane and consider the ring R of functions $a = a(z)$ on G such that $a(z) = O(|z|^\lambda)$ for some λ. Put $\varphi(a) = e^{\lambda_0}$, where λ_0 is the inf of these λ for the given a. Then φ is a non-archimedean pseudo-valuation on R, i.e.,

$$\varphi(ab) \leqq \varphi(a)\varphi(b) \quad\text{and}\quad \varphi(a-b) \leqq \max(\varphi(a), \varphi(b)).$$

Following Van der Corput, the author defines asymptotic convergence as convergence with respect to this pseudo-valuation, and proves many well-known properties of asymptotic expansions from this point of view. *J. Tate* (New York, N. Y.).

Fourier Series and Generalizations, Integral Transforms

Matsuyama, Noboru. On the methods of summability $(K, 1)$ and $(K, 2)$. Mem. Fac. Sci. Kyûsyû Univ. A. **6**, 113–120 (1952).

Zygmund's (K, n) summability $(n = 1, 2)$ arises out of the relation between a conjugate F.S. (Fourier series) and the $(C, n-1)$ means of the conjugate function, just as Riemann's (R, n) summability $(n = 1, 2)$ arises out of the relation between a F.S. and the (C, n) means of the function [Zygmund, Studia Math. **10**, 97–103 (1948); these Rev. **10**, 31]. The author's theorems are: 1) If (i) $\sum a_n$ is $(K, 2)$, (ii) $\sum n^{-1}a_n \cos nx$ is a F.S., then $\sum a_n$ is $(C, 2+\delta)$. 2) If (i) $\sum a_n$ is $(K, 1)$, (ii) $\sum a_n$ is $(K, 2)$, (iii) $\sum n^{-1}a_n \cos nx$ is a F.S., then $\sum a_n$ is $(C, 1+\delta)$. 3) If (i) $\sum a_n$ converges, (ii) $na_n > -C$, then $\sum a_n$ is $(K, 1)$. 4) If (i) $a_1 + \cdots + a_n = o(1/\log n)$, (ii) $n^2a_n > -C$, then $\sum a_n$ is $(K, 1)$. He states that 1) was obtained independently by Sunouchi [Tôhoku Math. J. (2), **1**, 167–185 (1950); these Rev. **12**, 174]. 3) was given substantially by Hardy and Littlewood [J. London Math. Soc. **1**, 19–25 (1926)]. Kuttner's analogue [Proc. London Math. Soc. (2) **38**, 273–283 (1934)] of 1) for $(R, 2)$ summability suggests that hypothesis 1) (ii) may be superfluous. Zygmund's analogue [Bull. Internat. Acad. Polon. Sci. Lett. Cl. Sci. Math. Nat. Sér. A. Sci. Math. **1924**, 243–249; cf. Hardy, Divergent series, Oxford, 1949, pp. 365–371; these Rev. **11**, 25] of 2) for $(R, 1)$ summability suggests that hypotheses 2) (ii) and 2) (iii) may be superfluous. Hardy and Littlewood's analogue [Ann. Scuola Norm. Super. Pisa (2) **3**, 43–62 (1934)] of 4 for $(R, 1)$ summability had $n^2a_n = O(1)$, the one-sided result being conjectured.

The author's arguments are confusing as he states a false lemma, which he attributes (without reference) to the reviewer. On inspection it appears that the lemma may nevertheless hold in the relevant cases, but it would be sufficient for his purpose to use the results: (a) if

$$(*)\qquad \lim_{t\to+0} \int_t^\pi \frac{\varphi(u)-\varphi(0)}{u^2}\,du$$

exists, then the conjugate derived F.S. of the even function $\varphi(t)$ is $(C, 2+\delta)$ at 0; (b) if $(*)$ exists and $\varphi(t) - \varphi(0) = o(t)$, then the series is $(C, 1+\delta)$ [cf. Bosanquet, Proc. London Math. Soc. (2) **49**, 63–76 (1945); these Rev. **7**, 154; together with Paley, Proc. Cambridge Philos. Soc. **26**, 173–203 (1930)]. *L. S. Bosanquet* (London).

Sunouchi, Gen-ichiro. A Fourier series which belongs to the class H diverges almost everywhere. Kōdai Math. Sem. Rep. **1953**, 27–28 (1953).

The author points out that the proof given by Hardy and Rogosinski [Fourier series, 2nd ed., Cambridge, 1950; these Rev. **13**, 457] of the existence of an integrable function whose Fourier series diverges almost everywhere, also shows the existence of a function of class H with the same property. *P. Civin* (Princeton, N. J.).

Ul'yanov, P. L. On trigonometric series with monotonically decreasing coefficients. Doklady Akad. Nauk SSSR (N.S.) **90**, 33–36 (1953). (Russian)

The author considers the functions $f(x) = \sum_{k=1}^\infty a_k \cos kx$, $\tilde{f}(x) = \sum_{k=1}^\infty a_k \sin kx$, under the condition that $a_k \to 0$ and $\sum|\Delta a_n| < \infty$. It is well known that neither series need be a Fourier series under this condition, if integration is Lebesgue integration. The author says that a measurable function $\phi(x)$ is A-integrable on (a, b) if the measure of the set where $|\phi(x)| \geqq n$ is $o(1/n)$ and the Lebesgue integral of the function obtained by truncating $\phi(x)$ at $\pm n$ approaches a limit. [See, e.g., Očan, Mat. Sbornik. N.S. **28(70)**, 293–336 (1951); these Rev. **13**, 20.] The following theorems are given. (1) The series for $f(x)$ and $\tilde{f}(x)$ are the A-Fourier series of their sums. (2) If $\phi(x)$ is a function of bounded variation whose conjugate $\tilde{\phi}(x)$ is also of bounded variation, $(A)\int_{-\pi}^{\pi} f\tilde{\phi} = -(A)\int_{-\pi}^{\pi} \tilde{f}\phi$. (3) If $\phi(x)$ is a bounded measurable function, the definition of $\tilde{\phi}(x)$ as a Cauchy integral agrees almost everywhere with the definition of $\tilde{\phi}(x)$ as an A-integral. (4) For all x, $\tilde{f}(x)$ is the negative of the A-conjugate of $f(x)$; except perhaps at $0, \pm\pi$, $f(x)$ is the A-conjugate of $\tilde{f}(x)$. *R. P. Boas, Jr.* (Evanston, Ill.).

Hammersley, J. M. A non-harmonic Fourier series. Acta Math. **89**, 243–260 (1953).

Let c be a complex number such that the equation

$$(1)\qquad \pi\lambda \cos \pi\lambda + c \sin \pi\lambda$$

has no double roots, and let λ_n $(n = 1, 2, 3, \cdots)$ denote the roots of (1) such that $-\pi/2 < \arg \lambda_n \leqq \pi/2$, indexed so that

$$\lambda_n = n - \tfrac{1}{2} + (c/\pi^2 n) + O(1/n^2).$$

If $-1 < c \neq 0$, then the λ_n are all real. If f is Lebesgue integrable on $(-\pi, \pi)$, consider the series

$$(2)\qquad \sum_{n=1}^{\infty} (a_n \cos \lambda_n x + b_n \sin \lambda_n x)$$

where

$$a_n = k_n \int_{-\pi}^{\pi} (\cos \lambda_n t - \cos \pi\lambda_n) f(t)\,dt, \quad b_n = k_n \int_{-\pi}^{\pi} f(t) \sin \lambda_n t\,dt,$$

$$k_n = (c^2 + \pi^2\lambda_n^2)/\pi[c(c+1) + \pi^2\lambda_n^2].$$

In the interior of $(-\pi, \pi)$, it is shown that (2) is summable to $f(x)$, in the Poisson-Abel sense, under the same conditions as the ordinary Fourier series of f. The results are obtained by equiconvergence methods; they fail to hold at the end-points. It is pointed out that the λ_n used here do not satisfy the condition which Levinson [Gap and density theorems, Amer. Math. Soc. Colloq. Publ., v. 26, New York, 1940; these Rev. **2**, 180] has proved to be "best possible" in a certain sense. *W. Rudin* (Rochester, N. Y.).

Gagliardo, Emilio. Sulla convergenza uniforme di alcune serie. Boll. Un. Mat. Ital. (3) **8**, 173–177 (1953).

Consider two sequences of numbers $\{a_n\}$, $\{\lambda_n\}$ with the properties that $\sum_n |a_{n+1} - a_n|$ converge and $\lambda_{n+1} - \lambda_n \to q > 0$

as $n\to\infty$. The author examines the properties of various series of the form $\sum_n a_n e^{\alpha_n x}$, $\sum_n a_n \cos \lambda_n x$, $\sum_n a_n \sin \lambda_n x$, $\sum_n (-1)^n a_n e^{\alpha_n x}$, determining intervals within which uniform convergence occurs. *R. Bellman* (Los Angeles, Calif.).

Stečkin, S. B. On absolute convergence of orthogonal series. I. Amer. Math. Soc. Translation no. **89**, 11 pp. (1953).

Translated from Mat. Sbornik N.S. **29**(71), 225–232 (1951); these Rev. **13**, 229.

Dževaršeĭšvili, A. G. On representation of a function by a Fourier integral. Soobščeniya Akad. Nauk Gruzin. SSR **13**, 201–205 (1952). (Russian)

This paper is concerned with the analog for Fourier integrals of a convergence criterion for Fourier series obtained earlier by the author [same Soobščeniya **11**, 403–407 (1950); these Rev. **14**, 635]. The present results, though derivable from those of the series case, are given separate treatment. The basic result is that if the integrable function $f(x)$ vanishes on a closed set E of positive measure on whose contiguous intervals $d_k f$ has oscillations $\omega(d_k; f)$, then the integral

$$S_\omega(x; f) = \pi^{-1}\int_{-\infty}^{\infty} f(x+t)t^{-1} \sin \omega t dt$$

converges to $f(x)$ at every point of density of E provided $\sum_{k=1}^{\infty}\omega(d_k; f) < +\infty$. *G. Klein* (South Hadley, Mass.).

Pollard, Harry. Distribution functions containing a Gaussian factor. Proc. Amer. Math. Soc. **4**, 578–582 (1953).

The problem of whether a distribution function $F(x)$ contains the normal distribution as a factor (in the sense of convolution) is the same as the problem of representing $F'(x) = f(x)$ in the form $\pi^{-\frac{1}{2}}\int_{-\infty}^{\infty} \exp[-(x-y)^2]d\alpha(y)$ with a distribution function $\alpha(y)$. The author establishes the following new necessary and sufficient condition: $f(x)$ belongs to C^∞, $\int_{-\infty}^{\infty} f(x)dx = 1$, and $\sum_{k=0}^{\infty}(-1)^k f^{(2k)}(x)(t/4)^k/k!$ converges to a nonnegative value uniformly on $0<x<\infty$, for each t, $0\leq t<1$. *R. P. Boas, Jr.* (Evanston, Ill.).

Pistoia, Angelo. Sulla operazione di composizione secondo una varietà lineare per la trasformata multipla di Laplace. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **15**(**84**), 241–249 (1951).

The multiple Laplace transform of a function F whose domain is the entire Euclidean N-dimensional space σ is defined as $f(p_1, \cdots, p_n) = \int_\sigma \exp(-\sum x_i p_i)F(x_1, \cdots, x_N)d\sigma$. A similar definition holds for the transform $g(\alpha_1, \cdots, \alpha_k)$ of a function G whose domain is the k-dimensional space τ, where $1\leq k\leq N$. Given a linear variety V_k through a point $(x_1, \cdots, x_n)$ in σ, the author defines the convolution $H = G*F$ over V_k by the formula

$$H(x_1, \cdots, x_n) = \int_\tau G(t_1, \cdots, t_k) \cdot F(x_1 - \sum t_i a_{i1}, \cdots, x_N - \sum t_i a_{iN})d\tau.$$

(This is a generalization of the convolution about an axis introduced, for $N=2$, by L. Amerio.) He then proves that if the integrals defining $f(p_1, \cdots, p_n)$ and $g(\sum p_i a_{1i}, \cdots, \sum p_i a_{ki})$ converge absolutely, that the transform over σ of $G*F$ is:

(1) $L(G*F) = f(p_1, \cdots, p_N)\cdot g(\sum p_i a_{1i}, \cdots, \sum p_i a_{ki})$.

Restricting his attention to the class C of functions F which vanish when any $x_i<0$ and functions G which vanish when any $t_s<0$, he shows that $G*F$ will be in C if and only if all $a_{si}\geq 0$. The transforms are then integrals taken over S: $x_1\geq 0, \cdots, x_n\geq 0$ or T: $t_1\geq 0, \cdots, t_k\geq 0$. Under the additional assumption that no row of the matrix $\|a_{si}\|$ consists only of zeros, the author proves the validity of formula (1) whenever $f(p_1, \cdots, p_N)$ converges boundedly and $g(\sum p_i a_{i1}, \cdots, \sum p_i a_{ki})$ converges absolutely.

D. L. Bernstein (Rochester, N. Y.).

Bose, S. K. Une chaîne de transformations de Laplace. Bull. Sci. Math. (2) **77**, 81–89 (1953).

This paper contains four results on operational calculus. First result: If $\phi(p) \subset f_1(t)$, $2^{1/2}f_1(p^{-1}) \subset f_2(t)$, and

$$\tfrac{1}{2}(\pi/p)^{1/2}f_k(1/4p^2) \subset f_{k+1}(t)$$

for $k=2, 3, \cdots, n-1$, then $\phi(p^{2^{n-1}}) \subset \frac{1}{2}\pi^{1/2}tf_n(t^2/4)$. The second result is a trivial consequence of the first one. Third result: if $\phi(p) \subset f_1(t)$, $p^{1/2}f_1(p^{-1}) \subset f_2(t)$, and $f_2(p) \subset g(t)$, then

$$\phi(p^2) = 3\pi^{1/2}p\int_0^\infty t^{-2} \exp\left(\frac{p^2}{2t}\right)D_{-4}\left[p\left(\frac{2}{t}\right)^{1/2}\right]g(t)dt.$$

The fourth result is the expansion of $f(x+y)$ in powers of y (this being, of course, Taylor's expansion, although the author seems to think that it is an alternative to Taylor's expansion). Each result is accompanied by a number of illustrative examples. *A. Erdélyi* (Pasadena, Calif.).

Bose, S. K. Some sequences of Laplace transforms. Bull. Calcutta Math. Soc. **44** (1952), 127–131 (1953).

Using the integral transformation $\phi(p) = pL\{h(x)\}$ where L denotes the Laplace transformation with parameter p, the author shows that if $\phi(p)$ is the transform of $x^\nu h(x)$, and $h(p)$ is in turn the transform of $f(x)$, and $p^\beta f(p^\mu)$ is the transform of $g(x)$, then $\phi(p)$ can be expressed in terms of an integral involving $g(x)$ and a specific but rather complicated function. Certain conditions on the complex constants ν, β, and μ and on the convergence of the integrals are assumed. A second theorem of similar character is presented; also one example of the first theorem. *R. V. Churchill.*

Colombo, Serge. Sur quelques transcendantes introduites par la résolution des équations intégrales de Volterra à noyaux logarithmiques. Bull. Sci. Math. (2) **77**, 89–104 (1953).

The author puts

$$T\{g(s)\} = \int_0^\infty \frac{x^s}{\Gamma(s+1)}g(s)ds$$

and shows that the T transforms of some simple functions are the functions $\nu(x)$, $\nu(x, a)$, $\mu(x, m)$, $\mu(x, m, n)$ which have been investigated by Humbert, Poli, and Colombo. He gives an extensive collection of formulas for these functions. The paper concludes with some remarks on the transformation

$$\mathcal{T}\{\phi(s)\} = \int_0^\infty \frac{\phi(s)ds}{\Gamma(x+s+1)}.$$

A. Erdélyi (Pasadena, Calif.).

Mikusiński, J. G. On the operational calculus. Zastosowania Mat. **1**, 28–40 (1953). (Polish. Russian and English summaries)

In this expository paper the author describes briefly Heaviside's operational calculus. He contends that the "realization" of the operational calculus by means of the Laplace transformation, while precise, changes the character of the calculus. He advocates the extension of the concept of numbers, functions, etc., to accommodate the entities of

operational calculus, and describes briefly the extension of the ring of convolution algebra to a field. The paper was written in 1950 and should not be taken as representing the ultimate achievement of its author in this field.

A. Erdélyi (Pasadena, Calif.).

Polynomials, Polynomial Approximations

Kreis, H. Über die Orthogonalpolynome. Mitt. Verein. Schweiz. Versich.-Math. **53**, 46–56 (1953).

For arbitrary abscissas $x_1, x_2, \cdots, x_n$, let

$$P_s(x) = x^s + a_{s-1}x^{s-1} + \cdots + a_0$$

be the corresponding orthogonal polynomials which satisfy the condition (1) $\sum_x P_h(x)P_k(x) = 0$, $h \neq k$. In particular, $P_0(x) = x^0 = 1$, $\sum_x P_0(x) = n \equiv N_0$; $\sum_x P_k(x) = 0$, for $k \neq 0$; $\sum_x x^h P_k(x) = 0$, for $h < k$. From this it follows that

$$P_1(x) = x - \sum_{i=1}^{n} x_i/n,$$

and for $n > 1$, the norm $N_1 \equiv \sum_x P_1^2(x) = \sum_x P_1(x) \neq 0$. The polynomial $P_2(x)$ can be determined from the equation $x^2 = P_2(x) + A_1P_1(x) + A_0P_0(x)$, since $A_0N_0 = \sum_x x^2P_0(x)$, $A_1N_1 = \sum_x x^2P_1(x)$, and $N_2 \equiv \sum_x P_2^2(x) = \sum_x x^2P_2(x) \neq 0, n > 2$. In this way the unique system of polynomials $P_3(x)$, $P_4(x), \cdots$, and norms $N_3, N_4, \cdots$ can be determined, and finally $P_n(x)$, which is identical to the product

$$P_n^*(x) \equiv (x-x_1)(x-x_2)\cdots(x-x_n) = P_n(x) + A_{n-1}P_{n-1}(x) + \cdots + A_0P_0(x).$$

Here $A_iN_i = \sum_x P_n^*(x)P_i(x) = 0$, $0 \leq i \leq n-1$, so that $A_i = 0$. Consequently

$$P_n(x) \equiv P_n^*(x) \quad \text{and} \quad N_n \equiv \sum_x P_n^2(x) = \sum_x P_n^{*2}(x) = 0.$$

If $z = x - m$, m arbitrary, the polynomials $Q_s(z) = P_s(x)$ likewise satisfy (1), and $N_s = \sum_{i=1}^n P_s^2(x_i) = \sum_{i=1}^n Q_s^2(z_i)$. If the x_i are the integers $1, 2, \cdots, n$, for $m = \frac{1}{2}(1+n)$, the z_i are an arithmetic sequence with $z_1 = -\frac{1}{2}(n-1)$ and $z_n = \frac{1}{2}(n-1)$. The polynomials $Q_s(z)$ and $(-1)^sQ_s(-z)$ satisfy (1), and $(-1)^sQ_s(-z) = Q_s(z)$. Between three consecutive polynomials $Q_s(z)$, the relation $Q_{s+1}(z) = zQ_s(z) - (N_s/N_{s-1})Q_{s-1}(z)$ holds. Furthermore,

$$N_s = \frac{s!s!}{\binom{2s}{s}}\binom{n+s}{2s+1}.$$

From these two formulas, the formula of Tchebycheff

$$Q_{s+1}(z) = zQ_s(z) - \frac{s^2(n^2-s^2)}{4(4s^2-1)}Q_{s-1}(z)$$

is obtained.

E. Frank (Evanston, Ill.).

Tricomi, Francesco G. Determinazione dei limiti per $n\to\infty$ degli estremi relativi dell'$n$esimo polinomio di Legendre. Boll. Un. Mat. Ital. (3) **8**, 107–109 (1953).

The author shows that the rth relative maximum of $|P_n(x)|$ approaches the rth relative maximum of $|J_0(x)|$, as $n\to\infty$. He also shows that this maximum occurs at $x = 1 - \frac{1}{2}n^{-2}j^2_{1,r} + O(n^{-4})$ where $j_{1,r}$ is the rth positive zero of $J_1(x)$.

A. Erdélyi (Pasadena, Calif.).

Toscano, Letterio. Sulle derivate dei polinomi di Laguerre e del tipo ultrasferico rispetto al parametro. Boll. Un. Mat. Ital. (3) **8**, 193–195 (1953).

Tricomi [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 227–233 (1952); these Rev. **14**, 270] has computed the derivative of a generalized Laguerre polynomial with respect to the parameter. The author gives an alternative form of Tricomi's result, and also a formula for the derivative, with respect to the parameter, of a constant multiple of an ultraspherical polynomial.

A. Erdélyi.

***T'ung, Ch'in-Mo, and Hsü, Hsien-Yü. On certain inequalities of the Turán type concerning Laguerre and ultraspherical polynomials.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 781–786. Akadémiai Kiadó, Budapest, 1952. (Hungarian and Russian summaries)

The authors prove the inequality

$$p_n^2(x) - p_{n-1}(x)p_{n+1}(x) \geq 0, \quad n \geq 1,$$

when (a) $p_n(x) = L_n^{(\alpha)}(x)$ (Laguerre polynomials), $\alpha > 0$, $x \geq 0$, (b) $p_n(x) = P_n^{(\lambda)}(x)$ (ultraspherical polynomials), $\lambda \geq 1/2$, $-1 \leq x \leq 1$. These results differ from inequalities proved by the reviewer [Bull. Amer. Math. Soc. **54**, 401–405 (1948); these Rev. **9**, 429]. The proof of (b) is based on induction in n. The notation follows the book of the reviewer [Orthogonal polynomials, Amer. Math. Soc. Colloq. Publ., v. 23, New York, 1939; these Rev. **1**, 14].

G. Szegö (Stanford, Calif.).

Koschmieder, Lothar. The boundary values of the derivatives of Gegenbauer polynomials. Univ. Nac. Tucumán. Revista A. **9**, 45–46 (1952). (Spanish)

The author computes the values of the derivatives of an ultraspherical polynomial at ± 1 from the generating function of these polynomials. (The result is implicit in equation (4.7.6) of Szegö's "Orthogonal polynomials", Amer. Math. Soc. Colloq. Publ., v. 23, New York, 1939; these Rev. **1**, 14.)

A. Erdélyi (Pasadena, Calif.).

Harmonic Functions, Potential Theory

Lelong-Ferrand, Jacqueline. Majoration de l'intégrale de Dirichlet dans certains espaces de Riemann. C. R. Acad. Sci. Paris **236**, 1227–1229 (1953).

Let $ds_0^2 = g_{ij}dx^idx^j$ be the metric of a compact Riemannian space V^n of class C^1. Let $f(x^1, \cdots, x^n)$ be a positive function of class C^1 in V^n and $\varphi(\rho)$ a positive function of class C^1 in the interval (a, b). Let $ds^2 = f^2d\varphi^2 + \epsilon\varphi^2ds_0^2$ $(\epsilon = \pm 1)$ be a metric on $V^{n+1} = (a, b) \times V^n$. As an extension of a result due to Beurling [Acta Math. **72**, 1–13 (1939); these Rev. **1**, 226], it is proved that

$$\int_{V^n}[\varphi^k(r)G(r, x^1, \cdots, x^n)\varphi^k(r_0)G(r_0, x^1, \cdots, x^n)]dx^1\cdots dx^n$$
$$= 2\int_{r_0}^{r}\int_{V^n}\varphi^k(\rho)\Delta u\frac{\partial u}{\partial\rho}fg^{\frac{1}{2}}d\rho dx^1\cdots dx^n$$
$$+ (k-2n)\int_{r_0}^{r}\int_{V^n}\varphi'\varphi^{k-1}\frac{g^{\frac{1}{2}}}{f}\left(\frac{\partial u}{\partial\rho}\right)^2 d\rho dx^1\cdots dx^n$$
$$-\epsilon(k-2)\int_{r_0}^{r}\int_{V^n}\varphi'\varphi^{k-3}fg^{\frac{1}{2}}g^{ij}\frac{\partial u}{\partial x^i}\frac{\partial u}{\partial x^j}d\rho dx^1\cdots dx^n,$$

where k is any real constant and

$$G=fg^{\frac{1}{2}}\left[\frac{1}{f^2}\left(\frac{\partial u}{\partial \rho}\right)^2-\frac{\epsilon}{\varphi^2}g^{ij}\frac{\partial u}{\partial x^i}\frac{\partial u}{\partial x^j}\right].$$

The formula is applied to some regularity properties of the function u of class C^2 in V^{n+1}. *K. Yosida* (Osaka).

Jongmans, F. Étude de géométrie kählérienne. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **39**, 77–93 (1953).

L'auteur établit plusieurs critères d'harmonicité pour les formes différentielles sur une variété kählérienne compacte V, de dimension m; par exemple: si une forme différentielle de degré $2k+p$ est fermée, pure et de classe k, elle est harmonique pourvu que $k<m-p$. L'espace vectoriel H des formes harmoniques sur V est muni d'une structure d'algèbre sur le corps des complexes, dans laquelle le produit de deux formes est la composante harmonique de leur produit extérieur. On considère les diviseurs stricts de zéro de l'algèbre H, c'est-à-dire ceux qui ne proviennent pas d'une décomposition triviale de zéro (produit de deux formes dont la somme des degrés est supérieure à $2m$ ou de deux formes de degré impair linéairement dépendantes). L'auteur donne, au moyen d'inégalités entre les nombres de Betti de V, des conditions d'existence des diviseurs stricts de zéro de H.

P. Dolbeault (Paris).

Spencer, D. C. Heat conduction on arbitrary Riemannian manifolds. Proc. Nat. Acad. Sci. U. S. A. **39**, 327–330 (1953).

Soit M une variété riemannienne orientable de classe C^∞. Sur M, on désigne par (φ, ψ) le produit scalaire $\int_M \varphi \wedge {}^*\psi$ de deux formes différentielles φ et ψ, et par $D(\varphi, \psi)$ le produit scalaire de Dirichlet $(d\varphi, d\psi)+(\delta\varphi, \delta\psi)$. Soient A^p l'espace des p-formes de norme finie, D^p l'adhérence de l'espace des p-formes φ de classe C^2 telles que φ, $d\varphi$, $\delta\varphi$ et $\Delta\varphi$ soient de normes finies. On pose: $G^p=\{\varphi | \varphi \in D^p, D(\varphi, \psi)=(\varphi, \Delta\psi)$ pour toute $\psi \in D^p\}$; $N^p=\{\varphi | \varphi \in D^p, D(\varphi, \psi)=(\Delta\varphi, \psi)$ pour toute $\psi \in D^p\}$ et $D(\varphi)=D(\varphi, \varphi)$; $D_s(\varphi, \psi)=D(\varphi, \psi)+s(\varphi, \psi)$; $\Delta_s=\Delta+s$. L'opérateur de Green G_s (resp. Neumann N_s) est l'application linéaire, biunivoque de A^p sur G^p (resp. N^p) dont l'inverse est Δ_s. Théorème. Pour tout $s>0$, les opérateurs de Green et de Neumann existent. Ils sont symétriques et satisfont à $D_s(G_s\varphi) \leq \|\varphi\|^2/s$, $D_s(N_s\varphi) \leq \|\varphi\|^2/s$. On déduit de N_s un opérateur W_t satisfaisant à $(\Delta+\partial/\partial t)W_t=0$ et, pour toute forme $\alpha \in A^p$, l'équation de la chaleur a la solution $W_t\alpha$ avec la distribution initiale α.

Si M est un domaine relativement compact d'une variété de Riemann R et si $F_0{}^p$ désigne l'espace des p-formes harmoniques sur R, identiquement nulles en dehors de M, il existe un opérateur de Green G_0 qui est une application linéaire complètement continue de A^p sur $G^p-F_0{}^p$, de noyau $F_0{}^p$, dont l'inverse est Δ. M est une u-variété (totally unbounded manifold) si toute p-forme appartenant à l'adhérence de l'espace des p-formes φ de classe C^2 telles que φ et $\Delta\varphi$ soient de norme finie, appartient aussi à N^p. Alors, $G^p=N^p$ et $G_s=N_s$ pour $s>0$; les formes harmoniques de norme finie sont fermées et cofermées et d, δ commutent avec N_s et W_t. Les variétés compactes sont des u-variétés. Les u-variétés sont à frontière négligeable [Gaffney, mêmes Proc. **37**, 48–50 (1951); ces Rev. **13**, 987]; mais il y a des variétés à frontière négligeables qui ne sont pas des u-variétés comme le montre un exemple. *P. Dolbeault* (Paris).

Bradford, W. H. Sub-biharmonic functions. Duke Math. J. **20**, 173–176 (1953).

L'auteur considère dans le plan les fonctions u douées de dérivées continues du 4ème ordre et donne des critères pour que $\Delta(\Delta u) \leq 0$. Par exemple il faut et suffit que, en chaque point M_0, et pour tout cercle de rayon ρ contenu dans le domaine de définition, u soit majorée par l'expression $(1-\mu^2)^{-1}(L_{\mu\rho}-\mu^2 L_\rho)$, où L_α désigne la moyenne (périphérique ou spatiale) de centre M_0 et rayon α. Croissance en ρ de l'expression précedénte. *M. Brelot* (Grenoble).

Ninomiya, Nobuyuki. Sur un ensemble de capacité nulle et l'infini d'un potentiel. Math. J. Okayama Univ. **2**, 99–101 (1953).

Tout ensemble borné $A \subset R^m$ qui est un G_δ de capacité extérieure nulle est l'ensemble des infinis d'un potentiel newtonien U^μ [J. Deny, C. R. Acad. Sci. Paris **224**, 524–525 (1947); ces Rev. **8**, 380]; si A est fermé, on peut imposer à la mesure positive μ d'être portée par A [G. C. Evans, Monatsh. Math. Phys. **43**, 419–424 (1936)]; l'auteur montre qu'il en est encore ainsi lorsque A est contenu dans un F_σ de capacité nulle; sa méthode est valable pour les potentiels d'ordre α, $0<\alpha<m$. *J. Deny* (Strasbourg).

Hodžaev, L. Š. A generalized Newtonian potential of bounded mass. Doklady Akad. Nauk SSSR (N.S.) **86**, 893–896 (1952). (Russian)

Let R_n be Euclidean n-space ($n \geq 3$) and define S_l as the space of real functions ψ on R_n which have continuous partial derivatives through order l and vanish outside corresponding bounded regions V_ψ. The author studies Poisson's equation $\Delta u=(2-n)[2\pi^{n/2}/\Gamma(n/2)]\rho$ on the space $S_l{}^*$ of linear functionals on S_l and shows that a generalized potential $u=P\rho$, defined in terms of an operator P on a subspace of $S_l{}^*$, yields a unique solution for suitably restricted ρ.

M. G. Arsove (Seattle, Wash.).

Pini, Bruno. Su certi integrali analoghi ai potenziali. Boll. Un. Mat. Ital. (3) **8**, 159–163 (1953).

Let $P(x_1, \cdots, x_n, y)$, $Q(\xi_1, \cdots, \xi_n, \eta)$ represent two points in the Euclidean space E_{n+1}, and V stand for a q-dimensional $(0 \leq q \leq n-1)$ variety in E_{n+1}. With P off the variety V, define $H(P)$ as $H(P)=\int_V [U(P, Q)]^a d_Q V$, where

$$U(P, Q)=(y-\eta)^{-n/2} \exp\left[-\sum_{i=1}^{n}(x_i-\xi_i)^2(y-\eta)^{-1}\right]$$

for $y>\eta$ and $U(P, Q)=0$ for $y \leq \eta$. Singular properties of $H(P)$ are studied in the present paper. Typical of the results is the following. Under certain regularity conditions on V, if V belongs to the hyperplane $y=a$, then

$$H(P)=f(P)\cdot(y-a)^{-(an-q)/2} \exp\left[-a\sum_{i=1}^{n}x_i^2/4(y-a)\right]+\phi(P).$$

Here $f(P)$ and $\phi(P)$ are bounded functions.

F. G. Dressel (Durham, N. C.).

***Gross, Wolf. Calcolo dell'attrazione newtoniana tra due dischi omogenei coassiali.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 507–515. Casa Editrice Perrella, Roma, 1953.

Computation of the mutual Newtonian attractive force between two solid coaxial cylinders leads to integrals which are not elementary. In the present note the author transforms these integrals into forms expressible in terms of standard tabulated elliptic integrals. This is done by show-

ing that the functions to be calculated are the solutions of certain linear differential equations, and then solving the equations. If the two cylinders have equal radii, the results assume a particularly simple form for which series expansions are obtained. *J. W. Green* (Los Angeles, Calif.).

Kac, M. An application of probability theory to the study of Laplace's equation. Ann. Soc. Polon. Math. 25 (1952), 122–130 (1953).

Let Ω be a bounded closed three-dimensional region and Ω_0 its interior. Then the Green's function $G(y, r)$ of the Laplace equation $\Delta_y u=0$ which vanishes at every regular point of the boundary is given by the formula

$$G(y, r)=\lim_{u\uparrow\infty}\lim_{\delta\to 0}\left(\frac{4}{3}\pi\delta^3\right)^{-1} \times\int_0^\infty E\left(\exp\left(-u\int_0^t V_B(y+r(\tau))d\tau\right),\ |y+r(t)-r|<\delta\right)dt.$$

Here $V_B(r)$ is the characteristic function of the set B which is the complement of Ω_0 with respect to any closed sphere containing Ω in its interior. And the expression $E(\cdots, \cdots)$ denotes the Wiener integral of $\exp[-u\int_0^t V(y+r(\tau))d\tau]$ over the set of those paths $r(t)$ which satisfy the condition $|y+r(t)-\gamma|<\delta$. The proof relies upon the author's result in the Proc. 2nd Berkeley Symposium on Math. Statistics and Probability [1950, Univ. of California Press, 1951, pp. 189–215; these Rev. 13, 568], and, as a by-product of the method of proof, it is proved probabilistically that

$$G(y, r)=(2\pi|y-r|)^{-1}-\lim_{u\uparrow\infty}\sum_{j=1}^{\infty}(u^{-1}+\mu_j)^{-1}(2\pi)^{-1} \times\int_B|\rho-y|^{-1}\psi_j(\rho)d\rho(2\pi)^{-1}\int_B|\rho-r|^{-1}\psi_j(\rho)d\rho,$$

where the μ_j's are the eigenvalues and the ψ_j's the normalized eigenfunctions of the integral equation

$$(2\pi)^{-1}\int_B|\rho-r|^{-1}\psi(\rho)d\rho=\mu\psi(r).$$

The results may be extended to higher dimensions but the method breaks down for the plane. *K. Yosida.*

Emersleben, Otto. Über das Restglied der Gitterenergieentwicklung neutraler Ionengitter. Math. Nachr. 9, 221–234 (1953).

This paper presents a concise review of the field indicated in the title. The author discusses the methods which have been developed for the computation of the energy of infinite neutral lattices (i.e., lattices which consist of an infinity of points). He then turns to finite lattices, showing in particular that the energy of such lattices may be expanded in descending powers of a parameter which depends on the number of lattice points. *A. Erdélyi.*

Differential Equations

Röhrl, Helmut. Über Differentialsysteme, welche aus multiplikativen, Klassen mit exponentiellen Singularitäten entspringen. III. Math. Ann. 125, 448–466 (1953).

In the first two parts of this memoir [Math. Ann. 123, 53–75 (1951); 124, 187–218 (1952); these Rev. 13, 224, 637] the author investigated the class k_B of functions y which may be briefly described as being of the form $R(z)W(z)$ where

$$W(z)=\prod_{\nu=1}^{n}(z-a_\nu)^{\alpha_\nu}e^{r(z)}$$

the a_ν, α_ν are fixed (given) constants, $r(z)$ is a fixed (given) rational function, and $R(z)$ is any (variable) rational function whose poles are at some of the points of the fixed (given, finite) set B. He investigated the complementary class $\mathring{k}_B$ of functions of the form $R(z)/W(z)$, and also integrals of functions of k_B and $\mathring{k}_B$. Integrals over closed paths are called periods. If $\{y_r\}$ is a (linear) basis for k_B, and $\{C_s\}$ is a basis for closed paths for $\int y dz$, then the periods of elements of k_B may be characterized in terms of the matrix W whose elements are $w_{sr}=\int_{C_s}y_r dz$. For the complementary class $\mathring{k}_B$ we have the period matrix $\mathring{W}$, and the author has shown that bases for functions and paths in k_B and $\mathring{k}_B$ may be chosen in such a manner that $W\mathring{W}'=I$, where a prime indicates transposition and I is the unit matrix.

In the present paper the author assumes that a_ν, $r(z)$, and also $R(z)$ depend on a complex parameter u so that the periods become functions of u. Roughly speaking, he starts with a differential field F of analytic functions of u. The coefficients of $r(z)$ are elements of F, and the $a_\nu=z_\nu(u)$ are zeros of coprime irreducible polynomials $f_\nu(z, u)$ in z, the coefficients of these polynomials being again elements of F. He then forms a differential field $\hat{F}$ which contains F, the $z_\nu(u)$, and the coefficients of $r(z)$. The $R(z)$ are then rational functions of z whose coefficients are elements of $\hat{F}$. The aggregate of all functions RW is shown to form a class of analytic functions of z, with $\hat{F}$ as the coefficient field. The class is closed under differentiation with respect to z or u; in particular, if $\{y_r\}$ is a basis for this class we have relations of the form $\partial y_s/\partial u=\sum_r d_{rs}y_r$, where the d_{rs} are elements of $\hat{F}$ and hence analytic functions of u: the matrix $[d_{rs}]$ is called D.

Under the present circumstances W is an analytic matrix function of u and the author shows that W satisfies the matrix differential equation $dW/du=WD$ so that the rows of W give integral representations of a fundamental system of solution vectors of a system of ordinary linear differential equations. The author discusses additional assumptions on F which will ensure that the coefficients of the system of differential equations are single-valued functions of u, and he discusses also the monodromic group of the system. He proceeds to the discussion of the complementary class, the discussion culminating in the result that complementary classes possess adjoint systems of differential equations.

The paper concludes with three examples of differential systems which are generalizations of the differential equations of, respectively, Hermite polynomials, Bessel functions, and confluent hypergeometric functions.

A. Erdélyi (Pasadena, Calif.).

***Gauthier, Luc. Au sujet de la recherche des cycles limites.** Actes du Colloque International des Vibrations non linéaires, Ile de Porquerolles, 1951, pp. 257–259, Publ. Sci. Tech. Ministère de l'Air, Paris, no. 281 (1953).

***Fichera, Gaetano. Interpretazione ed estensione funzionale di recenti metodi d'integrazione delle equazioni differenziali lineari.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. I, pp. 45–67. Casa Editrice Perrella, Roma, 1953.

This paper is a valuable synopsis of certain existence and uniqueness theorems (and applications) that have appeared

in recent years in the Italian literature, with special reference to those results connecting existence and (or) uniqueness with the completeness of appropriate systems of functions in appropriate spaces. *F. A. Ficken.*

Štokalo, I. Z. On the form of solutions of certain classes of linear differential equations with variable coefficients. Ukrain. Mat. Žurnal 4, 36–48 (1952). (Russian)

Under certain conditions the vector differential equation $x'-A(t)x=ce^{pt}$ has a solution $x=\omega(t,p)e^{pt}$ where $|\omega(t,p)|$ is bounded in t, $-\infty<t<\infty$, and analytic in p for $|\operatorname{Re} p|\geqq L$. Here c is a constant vector. In particular, if the matrix $A(t)$ satisfies $|A(t)|\leqq N$, then an L can be found depending on N so that ω exists. The author then concerns himself with the matrix equation $x'-A(t)x=F(t)$ where F is a matrix. If $\Psi(p)=\int_0^\infty e^{-p\tau}F(\tau)d\tau$ a solution of the form

$$x(t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}e^{pt}\omega(t,p)\Psi(p)dp$$

is shown, under certain conditions, to exist.

N. Levinson (Cambridge, Mass.).

Tartakovskiĭ, V. A. Explicit formulas for the solution of systems of ordinary differential equations. Ukrain. Mat. Žurnal 3, 128–160 (1951). (Russian)

In two earlier notes [Doklady Akad. Nauk SSSR (N.S.) **72**, 633–636, 853–856 (1950); these Rev. **12**, 27, 180] the author announced and slightly elaborated his new approach to the solution of the n-dimensional analytical system $\dot{x}=f(x)$ by infinite vectors and matrices. In this paper this work is elaborated at length. In particular, the Cauchy existence theorem is dispensed with and is obtained by means of direct estimates. [Additional reference: Lappo-Danilevskiĭ, Theory of functions of matrices . . . , GTII, Moscow, 1934.] *S. Lefschetz* (Princeton, N. J.).

***Richard, Ubaldo. Su una classe di "funzioni ausiliarie" riguardanti le equazioni differenziali del 2° ordine.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 200–203. Casa Editrice Perrella, Roma, 1953.

Summary of results taken from the author's articles in Univ. e Politecnico Torino. Rend. Sem. Mat. **9**, 309–324 (1950); **10**, 305–324 (1951); and in Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 382–387 (1952) [these Rev. **12**, 610; **13**, 653; **14**, 274]. *W. Wasow.*

Burdina, V. I. A criterion of boundedness of solutions of a system of differential equations of 2d order with periodic coefficients. Doklady Akad. Nauk SSSR (N.S.) **90**, 329–332 (1953). (Russian)

The author derives a criterion for the boundedness of the solutions of the second-order system $u'(t)=p_{11}(t)u+p_{12}(t)v$, $v'(t)=p_{21}(t)u+p_{22}(t)v$, where the $p_{ij}(t)$ are periodic functions; it is too complicated to state briefly.

R. Bellman (Santa Monica, Calif.).

***Sansone, Giovanni. Le equazioni delle oscillazioni non lineari—risultati analitici.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. I, pp. 186–217. Casa Editrice Perrella, Roma, 1953.

This is a useful summary of much of the more important work that has been done on the theory of non-linear oscillations since about 1920. The principal theorems are carefully stated, and in many cases indications of the proofs are given.

L. A. MacColl (New York, N. Y.).

***Graffi, Dario. Equazioni delle oscillazioni non-lineari in relazione alle applicazioni.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. I, pp. 218–231. Casa Editrice Perrella, Roma, 1953.

In this expository paper the author gives a non-technical survey of the present state of the theory of non-linear oscillations, and points out some of the physical applications of the results. *L. A. MacColl* (New York, N. Y.).

***Haag, Jules. Les oscillations non linéaires en chronométrie.** Actes du Colloque International des Vibrations non linéaires, Ile de Porquerolles, 1951, pp. 1–16; discussion, p. 15, Publ. Sci. Tech. Ministère de l'Air, Paris, no. 281 (1953).

In this expository article the author reviews recent work on the theory of oscillations of non-linear systems, particularly those parts of the subject that are of interest in connection with the design of clocks and other oscillating mechanisms. A large number of problems concerning stability of oscillations, synchronization, and the effects of dissipative forces are discussed briefly. *L. A. MacColl.*

***Mazet, R. Sur l'application de la méthode globale à l'étude de certains systèmes non linéaires.** Actes du Colloque International des Vibrations non linéaires, Ile de Porquerolles, 1951, pp. 17–18; discussion, pp. 19–20, Publ. Sci. Tech. Ministère de l'Air, Paris, no. 281 (1953). (French and English)

The author discusses the situation in which a differential equation of a prescribed form is assumed to govern a process, and in which one seeks to determine the constants and functions in the equation so that the solutions will agree with experiment. It is suggested that this can best be done by making use of periodic solutions. The development of the ideas is so meager that no definite results emerge.

L. A. MacColl (New York, N. Y.).

Dorodnicyn, A. A. Asymptotic solution of van der Pol's equation. Amer. Math. Soc. Translation no. **88**, 24 pp. (1953).

Dorodnicyn [Dorodnitsyn], A. A. Asymptotic solution of Van der Pol's equation. Translated by C. D. Benster. U. S. Department of Commerce, National Bureau of Standards, Los Angeles, Calif., NBS Rep. **2489**, 32 pp. (1953).

Translated from Akad. Nauk SSSR. Prikl. Mat. Meh. **11**, 313–328 (1947); these Rev. **9**, **144**.

Potter, Ruth Lind. On self-adjoint differential equations of second order. Pacific J. Math. 3, 467–491 (1953).

In the first part of the paper, some criteria for the differential equation (1) $y''+h^{-2}(x)y=0$, where $h>0$, to be oscillatory on $0\leqq x<\infty$ are derived. For example, if $\int^x(h^{-1}-h'^2/4h-h''/2)dx\to\infty$ as $x\to\infty$, then (1) is oscillatory, or if $\int^x(h^{-1}-h'^2/4h+h''/2)dx\to\infty$ as $x\to\infty$, then (1) is oscillatory if and only if $\int^\infty h^{-1}dx=\infty$. The paper also contains some theorems on the differential equation (2) $(r(x)y')'+p(x)y=0$ obtained from standard results on (1) by the obvious change of independent variables $x\to t$, defined by $dt=dx/r(x)$. The second half of the paper considers the boundedness of solutions of (2), where $r>0$ and either $p\leqq0$ or $p\geqq0$. Theorem 3.5, which deals with the case $p\leqq0$, is contained in much stronger known results. (A theorem of A. Kneser [J. Reine Angew. Math. **116**, 178–212 (1896)] states that $p\leqq0$ implies the existence of a solution satisfying $y>0$ and $y'\leqq0$ [for a short proof, cf. Hartman, Duke Math.

J. 15, 697–709 (1948), p. 702; these Rev. 10, 376]; if $r\equiv 1$, a necessary and sufficient condition for $y(\infty)=0$ is that $-\int^\infty xp(x)dx=\infty$. For the existence of such a solution, the author assumes that $r\equiv 1$ and that $-p>0$ is monotone increasing.) The main result when $p\geq 0$ states that if (2) is non-oscillatory, then all solutions are bounded (hence, have limits $y(\infty)$) if and only if $\int^\infty r^{-1}dx<\infty$. (This is curious and contrasts with a variant of a theorem of Wintner [Duke Math. J. 15, 55–67 (1948); these Rev. 9, 509; cf. Weyl, Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. 1909, 37–63, esp. p. 40, where $p\leq 0$] which implies that if $p(x)\not\equiv 0$ does not change its sign, then a necessary and sufficient condition that every solution have a limit $y(\infty)$ and that $y(\infty)\neq 0$ for some solution is that $\int^\infty r^{-1}(x)\int^x|p(t)|dtdx<\infty$.)

P. Hartman (Baltimore, Md.).

Gel'fand, I. M., and Levitan, B. M. On a simple identity for the characteristic values of a differential operator of the second order. Doklady Akad. Nauk SSSR (N.S.) 88, 593–596 (1953). (Russian)

Let $\{\lambda_n\}$ be the characteristic values of $-y''+q(x)y=\lambda y$, $y'(0)-hy(0)=0$, $y'(\pi)+Hy(\pi)=0$ where $\int_0^\pi q dx=0$. Let μ_n be the characteristic values of $y''+\mu y=0$ with same boundary conditions. The authors prove

$$\sum_{n=1}^{\infty}(\lambda_n-\mu_n)=\tfrac{1}{4}[q(0)+q(\pi)]+hH.$$

N. Levinson (Cambridge, Mass.).

Laasonen, Pentti. Über die iterative Bestimmung von Eigenwerten simultaner Differentialgleichungen. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 133, 14 pp. (1952).

The definite self-adjoint eigenvalue problem of Bliss, $u(x)=(F(x)+\lambda G(x))u(x)$, $Au(a)+Bu(b)=0$, $a\leq x\leq b$, with $u(x)$ as a column-vector with n components and with F, G, A, B as n by n matrices, is transformed into the homogeneous integral-equation $u(x)=\lambda\int_a^b K(x,y)u(y)dy$. Here $K(x,y)=\Gamma(x,y)G(y)$ where $\Gamma(x,y)$ denotes Green's matrix. Furthermore

$$K(x,y)=K^*(x,y)S(y)\quad\text{and}\quad S(x)K(x,y)=K'(y,x)S(y)$$

with a real, symmetrical, positive definite or positive semidefinite matrix $S(x)$. Here K' means the transposed matrix of K; $K^*(x,y)$ is a suitable matrix, which is continuous in the triangles $a\leq x\leq y\leq b$ and $a\leq y\leq x\leq b$; $S(x)$ is continuous for $a\leq x\leq b$. The author applies the results of his paper on an integral equation of this type [see same Ann. no. 118 (1952); these Rev. 15, 37] and obtains in this manner the well-known theorems of Bliss. He also describes an iterative method for the evaluation of eigenvalues (using the well known Schwarz constants) and concludes with some other remarks on iteration.

H. Bückner.

Thomas, Johannes. Ein Satz über die zur Eigenwertaufgabe des linearen Turbulenzproblems gehörige Greensche Funktion. Nachtrag zu meiner Arbeit: Untersuchungen über das Eigenwertproblem

$$\frac{d}{dx}\left(f(x)\frac{dy}{dx}\right)+\lambda g(x)y=0;\quad \int_a^b A(x)ydx=\int_a^b B(x)ydx=0$$

(Diese Nachr. 6, 229–260 (1951)). Math. Nachr. 9, 379–383 (1953).

Green's function $G(x,\xi)$ for the differential expression $d^2z/dx^2-\alpha^2 z$ (α a constant) with the boundary conditions $\int_0^1 e^{-\alpha x}z\,dx=0$, $\int_0^1 e^{\alpha x}z\,dx=0$, is calculated explicitly with the help of a theorem from the article referred to in the title [these Rev. 13, 655]. It is then shown that the function $G(x,x)$ has the following properties: (a) $G(x,x)=G(1-x,1-x)$; (b) $G(0,0)=(dG/dx)_{x=0}=(dG/dx)_{x=1/2}=0$; (c) $dG/dx<0$, in $0\leq x\leq 1/2$; (d) there is exactly one inflection point of $G(x,x)$ in $0<x<1/2$.

W. Wasow (Los Angeles, Calif.).

Furuya, Shigeru. Note on a boundary value problem. Comment. Math. Univ. St. Paul. 1, 81–83 (1953).

H. Weyl [Ann. of Math. (2) 43, 381–407 (1943); these Rev. 3, 284] proved the existence of a solution of the problem

$$w'''+2ww''+2\lambda(k^2-w'^2)=0,\quad\text{for}\quad z\geq 0;$$
$$w(0)=w'(0)=0,\quad w'(z)\to k,\quad\text{for}\quad z\to\infty.$$

In the present paper it is shown that the solution is unique, provided $0\leq\lambda\leq 1$. Assuming first that $w''(z)>0$, the uniqueness is shown in a few lines by a consideration of the transformed problem obtained by introducing $x=w(z)$ and $y=w'(z)$ as new variables. The proof of the hypothesis $w''(z)>0$ requires separate discussion for the ranges $0\leq\lambda\leq\frac{1}{2}$ and $\frac{1}{2}<\lambda<1$.

W. Wasow (Los Angeles, Calif.).

Harada, Shigeharu. An existence proof of the generalized Green function. Osaka Math. J. 5, 59–63 (1953).

The author is concerned with the details of computation of a generalized Green's function for a compatible self-adjoint boundary problem involving the differential equation $L(u)\equiv[p(x)u']'+q(x)u=0$ and two-point boundary conditions of either the form

(i) $\quad u(a)-\gamma u(b)=0,\quad \gamma p(a)u'(a)-p(b)u'(b)=0$

or the form (ii) $u(a)-\gamma p(b)u'(b)=0$, $\gamma p(a)u'(a)+u(b)=0$, where γ is a non-zero real number, while on $a\leq x\leq b$ the functions $q(x)$, $p(x)$ are continuous and $p(x)>0$. In the discussion it is assumed implicitly that the boundary problem has index of compatibility equal to one. Evidently the author was not familiar with much of the literature on generalized Green's functions [see W. W. Elliott, Amer. J. Math. 50, 243–258 (1928); 51, 397–416 (1929); and W. T. Reid, ibid. 53, 443–459 (1931), as well as references cited in these papers].

W. T. Reid (Evanston, Ill.).

Naimark, M. A. On expansion in characteristic functions of non-self-adjoint singular differential operators of the second order. Doklady Akad. Nauk SSSR (N.S.) 89, 213–216 (1953). (Russian)

The expansion theorem for $l(y)=-y''+p(x)y=\lambda y$, $y'(0)-\theta y(0)=0$ is considered over $0\leq x<\infty$ where θ and p are complex-valued. It is further assumed $\int_0^\infty x^2|p(x)|dx<\infty$. Solutions y_1 and $\bar{y}_1$ of $l(y)=s^2y$, $\lambda=s^2$, are set up in familiar fashion so that $y_1(x,s)\sim e^{isx}$, $\operatorname{Im}s\geq 0$, as $x\to\infty$ and $\bar{y}_1(x,s)\sim e^{-isx}$, $\operatorname{Im}s\leq 0$. Let $A(s)=y_1'(0,s)-\theta y_1(0,s)$, $\operatorname{Im}s\geq 0$, and similarly for $\bar{A}(s)$. Let $y(x,s)=\bar{A}(s)y_1-A(s)\bar{y}_1$ for $s\geq 0$. Let the discrete eigenvalues of the above problem be $\lambda_1,\cdots,\lambda_r$. Then the Green's function for all λ not in the spectrum is

$$K(x,\xi,\lambda)=\sum_{k=1}^{r}\frac{y_k(x)y_k(\xi)}{(\lambda_k-\lambda)\int_0^\infty y_k^2(x)dx}-\frac{1}{2\pi}\int_0^\infty\frac{y(x,s)y(\xi,s)}{(s^2-\lambda)A(s)\bar{A}(s)}ds.$$

The expansion theorem is also given.

N. Levinson.

Pipes, Louis A. An operational analysis of traffic dynamics. J. Appl. Phys. 24, 274–281 (1953).

The California Vehicle Code suggests: "A good rule for following another vehicle at a safe distance is to allow yourself the length of a car (about fifteen feet) for every ten miles per hour you are travelling." If each vehicle in a line of traffic trails the one immediately ahead by a distance linear in its own velocity v_k, the coordinate x_k of the kth vehicle is related to x_{k-1} by the equation

$$x_{k-1}=x_k+(b+Tv_k)+L_{k-1},$$

where L_k is the length of the kth vehicle, b is the standstill separation, and T is a constant to be determined by human reaction time measurements. In terms of the velocities v_k the derived differential equations are:

$$Tdv_k/dt+v_k=v_{k-1}, \quad k=1, 2, 3, \cdots, n.$$

[See A. Reuschel, Österreich. Ing.-Arch. 4, 193–215 (1950).] The author derives the system and applies Laplace transform theory, obtaining a compact derivation of the general solution. He solves explicitly three special cases, characterized by the motion of the lead vehicle: (a) impulsive starts and stops, (b) exponential acceleration and decleration, (c) constant acceleration.

Denoting the incomplete gamma function by

$$Q_k(t)=\int_0^t e^{-u}u^{k-1}du$$

and putting $G_k(t)=Q_k(t)/Q_k(\infty)$ the author expresses the velocities as (a) $v_k=v_0G_{k-1}(t)$ and $v_k=v_0(1-G_{k-1}(t))$, (b) $v_k=v_0G_k(t)$, and $v_k=v_0(1-G_k(t))$. In case (c) the velocities are integrals of the G_k. Graphs, and a short table of values, of the G_k are given for $k=1, 2, \cdots, 6$.

The author notes that for $T=1$ second, and $v_0=50$ miles/hour, a crash stop of the lead vehicle requires initial deceleration of 73.4 ft/sec² on the part of the second vehicle. The behavior code is seen to be impractical at ordinary road speeds.

The reviewer suggests that studies of quadratic separation laws, and of the system induced by delayed driver responses, would be valuable from the Operations Research viewpoint.
A. A. Brown (Wellesley, Mass.).

Bouligand, Georges. Sur un type d'énoncé stable en théorie des transformations de contact. C. R. Acad. Sci. Paris 236, 2136–2138 (1953).

L'auteur reprend un théorème d'aplatissement donné dans une note antérieure [mêmes C. R. 236, 1217–1219 (1953); ces Rev. 14, 900]. Il signale que l'indépendance des vecteurs $\partial M/\partial x+p\partial M/\partial z$, $\partial M/\partial y+q\partial M/\partial z$, en assurant la biunivocité locale entre S et son image, est une hypothèse essentielle à la stabilité de l'aplatissement. *Chr. Pauc.*

Bouligand, Georges. Sur quelques types d'équations $f(x, y, z, p, q)=0$. C. R. Acad. Sci. Paris 236, 2193–2195 (1953).

L'auteur considère les équations aux dérivées partielles du premier ordre dont le cône des normales est en chaque point du second dégré et capable de trièdres trirectangles inscrits. Le champ des cônes est déterminé de manière unique si on impose à l'équation d'avoir comme solutions les surfaces des familles de deux systèmes triples orthogonaux (S) et (P). (P) en particulier peut être un système de plans.
Chr. Pauc (Nantes).

Bouligand, Georges. Une forme donnée à la recherche des systèmes triples orthogonaux. C. R. Acad. Sci. Paris 236, 2462–2463 (1953).

(Continuation de l'analyse précédente.) Deux choix simples de (P) sont envisagés, le système des plans de coordonnées $x=c_1$, $y=c_2$, $z=c_3$ et le système des plans $x-y=d_1$, $x+y=d_2$, $z=d_3$. Ainsi sont adjointes à (S) deux èquations aux dèrivèes partielles. Une condition d'existence d'intégrales communes est donnée. *Chr. Pauc.*

***Haimovici, Mendel. Sur l'intégration des systèmes de deux équations aux dérivées partielles du I-er ordre à deux fonctions inconnues de deux variables indépendantes.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 585–590. Akadémiai Kiadó, Budapest, 1952. (Hungarian and Russian summaries)

Es handelt sich um die Verallgemeinerung der Darbouxschen Integrationsmethode auf Systeme zweier partieller Differentialgleichungen in zwei abhängigen und zwei unabhängigen Veränderlichen. Die Werte der vier ersten Ableitungen der beiden unbekannten Funktionen u und v nach den unabhängigen Veränderlichen x und y erscheinen als Koeffizienten eines mit dem gegebenen System äquivalenten Pfaffschen System zweier Gleichungen in vier Veränderlichen. Jedem Integrallinienelement E_1 des Pfaffschen Systems entspricht ein zweidimensionales Integralelement E_2 gebildet aus E_1 und einem zweiten zu E_1 involutorischen Integrallinienelement. Aus dem Verschwinden der bilinearen Kovarianten der Pfaffschen Gleichungen gewinnt der Verfasser zwei weitere Pfaffsche Gleichungen, deren Koeffizientenmatrix für nichtcharakteristische Richtungen den Rang 2 hat. Für charakteristische Richtungen vermindert sich dieser Rang. Diese Rangeigenschaften liegen der weiteren Entwicklung der Integrationstheorie zu Grunde. *M. Pinl* (Dacca).

Fourès-Bruhat, Yvonne. Les distributions sur les multiplicités. C. R. Acad. Sci. Paris 236, 2201–2202 (1953).

L'étude des équations hyperboliques exige la définition sur une multiplicité de distributions ayant les propriétés suivantes: les fonctions sont des distributions, une mesure est le produit d'une distribution par un élément de volume, les lois de dérivation pour les fonctions et les distributions sont les mêmes. Nous donnons ici une telle définition et une application. (Author's summary.) *F. John.*

Volkov, D. M. On exact solutions of a class of hyperbolic equations having application in the theory of hydraulic shock. Doklady Akad. Nauk SSSR (N.S.) 90, 49–50 (1953). (Russian)

Certain problems of hydraulic shock in tubes of variable cross-section lead to the equation

$$\frac{\partial^2 u}{\partial\xi\partial\eta}=B(\xi+\eta)\left(\frac{\partial u}{\partial\xi}+\frac{\partial u}{\partial\eta}\right).$$

By a separation of variables of the type $\sum_\nu g_\nu(\xi+\eta)d^\nu\varphi(\xi)/d\xi^\nu$ certain exact solutions of the above equation can be obtained. *M. H. Protter* (Berkeley, Calif.).

Ludford, G. S. S. Sur une difficulté dans la méthode de Riemann. C. R. Acad. Sci. Paris 236, 2293–2295 (1953).

The solution $u(x, y)$ of $u_{xy}+au_x+bu_y+cu=0$ with prescribed Cauchy data on a curve C is given by a well known integral formula, which involves the data and the Riemann

function on C. This formula actually represents a solution of the Cauchy problem only as long as C has no characteristic direction in any of its points. The author observes that even in the case where C is characteristic at one point P the formula still represents a solution of the Cauchy problem, which then however has to be interpreted as a multiple-valued function defined on two sheets, which come together along the tangent to C at P. *F. John.*

***Bureau, Florent.** **Problème de Cauchy et problème aux limites pour les équations linéaires aux dérivées partielles totalement hyperboliques.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 24–29. Casa Editrice Perrella, Roma, 1953.

The solution of the problem

$$u_{tt}-u_{xx}-a(x)u=0, \quad u(0,t)=0=u(l,t),$$
$$u(x,0)=0, \quad u_t(x,0)=u_1(x),$$

is

$$u(x,t)=\sum_{k=1}^{\infty}\int_0^l u_1(y)v_k(x)v_k(y)\frac{\sin \lambda_k^{\frac{1}{2}}t}{\lambda_k^{\frac{1}{2}}}dy,$$

where λ_k ($\lambda_1>0$) are the eigenvalues and $v_k(x)$ the corresponding eigenfunctions of the problem

$$\text{(Q)} \qquad v''+(\lambda+a(x))v=0, \quad v(0)=0=v(l).$$

On the other hand, inside a triangle ABP bounded by a segment AB of the initial line and the characteristics through A and B, u solves a Cauchy problem and can be expressed as an integral involving a suitable Riemann function R. The object of this paper is to compare the two solutions.

Let $G(x,y,\lambda)=\sum_{r=0}^{\infty}G_r(x,y)\lambda^r$ be the Green's function for (Q), let

$$H_s(x,y;t)=\sum_{k=1}^{\infty}\frac{v_k(x)v_k(y)}{\lambda_k^s}\frac{\sin \lambda_k^{\frac{1}{2}}t}{\lambda_k^{\frac{1}{2}}}$$

(s complex), let n be an integer $\geqq 1$, and define

$$M_n(x,y;t-\tau)=-H_n(x,y;t-\tau)$$
$$+(-1)^{n-1}\sum_{r=0}^{\infty}(-1)^r\frac{(t-\tau)^{2(n+r)-1}}{\Gamma(2n+2r)}G_r(x,y)$$

($n-r$ in the text for $n+r$). The principal result is that

$$u(x,t)=(-1)^n\frac{\partial^{2n}}{\partial t^{2n}}\int_0^l M_n(x,y;t)u_1(y)dy.$$

Properties of M_n are indicated which yield the desired comparison, and several relations between H, R, and $\zeta_s(x,y)=\partial H_s(x,y;0)/\partial t$ are stated. Detailed proofs are omitted. *F. A. Ficken* (Knoxville, Tenn.).

Pucci, Carlo. **Teoremi di esistenza e di unicità per il problema di Cauchy nella teoria delle equazioni lineari a derivate parziali. I.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **13**, 18–23 (1952).

Given the differential equation

$$\text{(1)} \qquad \frac{\partial^m u}{\partial t^m}-a\frac{\partial^n u}{\partial x^n}=f(x,t)$$

and the initial conditions

$$\text{(2)} \quad u(x,0)=u_0(x), \quad \left[\frac{\partial^i u}{\partial t^i}\right]_{t=0}=u_i(x), \quad i=1,2,\cdots,m-1,$$

where f is continuous in t on R: $x'\leqq x\leqq x''$, $|t|\leqq l$, and derivatives $(\partial^h F)/(\partial x^h)$, $(d^h u_i)/(dx^h)$ exist for all $h=1,2,3,\cdots$ and are bounded on R by $M\Gamma(mh/n+1)\rho^{-mh/n}$, the author proves that there exists a solution $u(x,t)$ of (1) and (2) on R, which is a function of class C, the class of functions with continuous derivatives with respect to x of all orders and with continuous derivatives with respect to t of orders $\leqq m$. The method of proof is to obtain a formal solution:

$$u(x,t)=\sum_{h=0}^{\infty}\sum_{i=0}^{m-1}a^h u_i^{(nh)}(x)\frac{t^{mh+i}}{(mh+i)!}$$
$$+\sum_{h=0}^{\infty}a^h\int_0^t\frac{(t-\tau)^{(h+1)m-1}}{[(h+1)m-1]!}\frac{\partial^{nh}f(x,\tau)}{\partial x^{nh}}d\tau,$$

prove that the series converge absolutely and uniformly in R and that the function u satisfies (1) and (2) and is of class C. He also shows that in R': $x'\leqq x\leqq x''$, $|t|\leqq l/3$, the solution satisfies inequalities of the form

$$\text{(3)} \qquad \left|\frac{\partial^s u(x,t)}{\partial x^s}\right|<L\Gamma\left(\frac{m}{n}s+1\right)(\rho/3)^{-ms/n}.$$

The reviewer suggests that the same problem could be solved by using a double Laplace transform.

D. L. Bernstein (Rochester, N. Y.).

Pucci, Carlo. **Teoremi di esistenza e di unicità per il problema di Cauchy nella teoria delle equazioni lineari a derivate paraziali. II.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **13**, 111–116 (1952).

Continuing his study of the Cauchy problem given in the paper reviewed above, the author proves that there exists at most one function $u(x,t)$ satisfying (1) and (2) in R and belonging to class A (the subclass of C consisting of those functions satisfying inequalities (3)). He then generalizes both the existence and the uniqueness theorems to functions defined on S: $x'\leqq x\leqq x''$; $-\infty<t<+\infty$. The paper concludes with a series of observations concerning the solution when $m<n$ (parabolic type), when $m=n$ (elliptic for $a<0$ and hyperbolic for $a>0$), and when $m>n$, indicating various special cases which have been studied by previous investigators. *D. L. Bernstein* (Rochester, N. Y.).

Pucci, Carlo. **Sul problema di Cauchy per le equazioni lineari a derivate parziali.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **14**, 198–202 (1953).

The author establishes the existence and uniqueness of a solution of the equation: $u_{tt}+a(t)u_{xx}=f(x,t)$ with the initial conditions: $u(x,0)=u_0(x)$, $u_t(x,0)=u_1(x)$, in a finite rectangle R, provided that f, u_i, and the desired solution u are restricted to a certain class of functions having derivatives with respect to x of all orders satisfying inequalities of the form $|\partial^n g/\partial x^n|<Mn!/\rho^n$. Defining an operator B by the formula $Bg(x,t)=-t*a(t)g_{xx}(x,t)$, where $*$ indicates the usual convolution operator, he shows that the solution can be written $u(x,t)=\sum B^n F(x,t)$, where

$$F(x,t)=t*f(x,t)+u_0(x)+u_1(x)t.$$

D. L. Bernstein (Rochester, N. Y.).

Nitsche, Johannes. **Über Unstetigkeiten in den Ableitungen von Lösungen quasilinearer hyperbolischer Differentialgleichungssysteme.** J. Rational Mech. Anal. **2**, 291–297 (1953).

Let $u(x^1,x^2)$, $v(x^1,x^2)$ denote a solution of a hyperbolic system of two quasi-linear first-order differential equations. The solution is said to have a discontinuity of order n along a curve K separating two regions G_1, G_2, if derivatives of u, v of order n (but not those of lower order) have a jump

discontinuity on K. Necessary for the occurrence of such a discontinuity with $n\geqq 1$ is that K is characteristic. It is shown that the magnitude of the jump (for u, v given in G_1) satisfies an ordinary differential equation along K, which is linear for $n>1$, but is a non-linear equation of the Riccati type for $n=1$. (This is in contrast to the situation for solutions u, v of a linear system of partial differential equations.) Thus prescribing the magnitude of the jump for a first-order discontinuity in one point of K may have as a consequence that the jump becomes infinite in some other point, and hence that continuation of the solution across all of K may be impossible. The formulae are applied to the equations of steady supersonic flow. *F. John.*

Lax, Peter D. Nonlinear hyperbolic equations. Comm. Pure Appl. Math. 6, 231–258 (1953).

This paper deals with the solution of a quasi-linear system of n equations in 2 independent variables and n unknown functions subject to given initial conditions. In matrix notation, it can be written: $U_y=A(x,y,U)U_x+C(x,y,U)$; $U(x,0)=\phi(x)$. The system is hyperbolic in a domain E of $(n+2)$-dimensional Euclidean space; that is, at every point of E, there is a non-singular matrix T such that $T^{-1}AT=D$, a real diagonal matrix. The author proves two main results: (1) If A, C, T, ϕ are analytic in E and U is an analytic solution of the problem in some domain of determinacy S such that U_x is bounded and such that the closure of the set of points $[x,y,U(x,y)]$ lies in the interior of E, then U has no singularities on the boundary of S. (2) When A, C, T, ϕ satisfy uniform Lipschitz conditions in a closed domain E', they can be approximated by uniformly convergent sequences of analytic functions; the approximating problems: $U_y=A_jU_x+C_j$; $U(x,0)=\phi_j(x)$, $(j=1,2,3,\cdots)$ have analytic solutions U_j which exist in a common domain of determinacy S' and converge uniformly in S' to a function U, called the generalized solution of the given problem.

In establishing these 2 theorems, the author proves other interesting results. For example, in order to prove (1), he obtains an a priori estimate for the solution of a diagonal linear system in terms of its initial values, and he proves certain inequalities for the majorants of power series in several variables which are presented in a concise form. In order to prove (2), he shows that for the analytic problem, if E' is a closed subdomain of E, an analytic solution can always be continued into a certain fixed domain S_y, which depends only on the bounds of the given functions and their first derivatives in E'. After establishing (2) he obtains some properties of generalized solutions, showing that with additional conditions on the coefficients, U becomes a solution in the usual sense, as previous investigators have shown, and that if the given functions have continuous first derivatives and ϕ has a continuous derivative almost everywhere, the first derivatives of the generalized solution are continuous at "regular" points. *D. L. Bernstein.*

Yosida, Kôsaku. On the fundamental solution of the parabolic equation in a Riemannian space. Osaka Math. J. 5, 65–74 (1953).

The author extends his previous treatment of the diffusion equation in a Riemannian metric [Proc. Amer. Math. Soc. 3, 864–873 (1952); these Rev. 14, 560] from a compact to a non-compact space under an hypothesis which roughly amounts to the following assumptions: the volume of a unit sphere (by distance) is bounded, and the (would-be) analogue $H(s,t;x,y)$ of the Weierstrass function

$$(s-t)^{-1/2}\exp[-(x-y)^2/(s-t)^{1/2}]$$

is such that $\int|H(s,t;x,y)|dv_x$ is finite, and bounded with regard to s, t, y in a certain fashion. *S. Bochner.*

Itô, Seizô. The fundamental solution of the parabolic equation in a differentiable manifold. Osaka Math. J. 5, 75–92 (1953).

This is a direct sequel to the paper reviewed above. The author dispenses with the assumption that there is given on the space a Riemannian metric which dove-tails into Yosida's hypothesis, and he constructs a parametrix in which only the (time-dependent) positive definite symmetric tensor $a^{ij}(t;x,y)$ of the equation itself is being used. Also his constructions lean rather closely on earlier ones by W. Feller and F. G. Dressel. On the whole the author's assumptions are rather more general, and more pertinent to the desired outcome than are Yosida's, but the assumptions of the latter are rather more easily grasped and assessed.

S. Bochner (Princeton, N. J.).

✶Pál, Sándor. Diffusion problems of the sugar industry. Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 693–726. Akadémiai Kiadó, Budapest, 1952. (Hungarian. Russian summary)

The mathematical part of this paper contains solutions of several special (one-dimensional or axi-symmetric) boundary-value problems of the diffusion equation.

A. Erdélyi (Pasadena, Calif.).

Nitsche, Johannes, und Nitsche, Joachim. Das zweite Randwertproblem der Differentialgleichung $\Delta u=e^u$. Arch. Math. 3, 460–464 (1952).

The differential equation of the title is to be solved in a simply connected domain T of the (x,y)-plane for a prescribed exterior normal derivative $f(s)$ (referred to the arc length s) of u on the boundary S of T. It is easily seen that the solution is unique and that a necessary condition for its existence is that (*) $\int_S f(s)ds>0$. It is proved that in case (*) holds the problem actually has a solution under the assumption that S is three times continuously differentiable and f satisfies a Hölder condition, provided T, $|f|$ and the Hölder constant for f are all sufficiently small. The proof makes use of the Green's function of T for the Neumann problem for the potential equation. With the help of that function the problem can be reduced to a non-linear integral equation (depending on a parameter to be determined from an additional integral relation), which can be solved by iteration. *F. John* (New York, N. Y.).

Šnol', I. È. On bounded solutions of a partial differential equation of the second order. Doklady Akad. Nauk SSSR (N.S.) 89, 411–413 (1953). (Russian)

The problem $\Delta u+(q-\lambda)u=0$ is considered in R_n where q is a function of $(x_1,\cdots,x_n)$. Let $q^+=\max(0,q)$. Let $r^2=x_1^2+\cdots+x_n^2$ and let $q^+=o(r^2)$ in R_n. The author proves that if u is a bounded solution of $\Delta u+qu=0$ then $\lambda=0$ is a point of the spectrum. If further $\int u^2dV$, over R_n, diverges, then $\lambda=0$ is a limit point of the spectrum. If q grows as fast as r^2, an example shows the result above to fail.

N. Levinson (Cambridge, Mass.).

Čudov, L. A. Isolated singular points and lines of solutions of linear partial differential equations. Doklady Akad. Nauk SSSR (N.S.) 90, 507–508 (1953). (Russian)

The solution $u(x,y,z)$ of the equation

$$(*)\qquad \sum_{0\leq i+j+k\leq n} A_{ijk}\frac{\partial^{i+j+k}u}{\partial x^i\partial y^j\partial z^k}=F(x,y,z)$$

is said to have an isolated singular point of bounded type at $P(x_0, y_0, z_0)$ if (i), in a deleted neighborhood of P, u has continuous derivatives of order $L(\geqq n)$; (ii) u satisfies (*) in this neighborhood; and (iii) there are constants A and p such that $|\partial^{n-1}u/\partial x^i\partial y^j\partial x^k| < A/r^p$ where

$$r^2=(x-x_0)^2+(y-y_0)^2+(z-z_0)^2.$$

The theorem is stated: If the non-degenerate equation

$$\sum_{i+j+k=n} A_{ijk}\frac{\partial^n u}{\partial x^i\partial y^j\partial z^k}=0,\quad A_{ijk}\text{ constants,}$$

has an isolated singular point with $L\geqq n+3$ then the equation is elliptic. A related theorem is stated concerning singular lines and hyperbolic equations. *M. H. Protter.*

Morgan, A. J. A. The reduction by one of the number of independent variables in some systems of partial differential equations. Quart. J. Math., Oxford Ser. (2) **3**, 250–259 (1952).

A system of kth order partial differential equations $\Phi_s=0$, in m independent variables x^i and n dependent variables y_k, where $m\geqq 2$ and $n\geqq 1$, is called invariant under a continuous one-parameter group of transformations G_1:

$$\bar{x}^i=f^i(x^1,\cdots,x^m;a),\quad \bar{y}_k=f_k(y_k;a)$$

if each of the kth order differential forms Φ_s is conformally invariant under the kth enlargement of the group G_1. (One obtains this enlargement by assuming the y_k's are differentiable functions, up to order k, of the x^i's, and adding to the equations of G_1, the transformation equations of all of these partial derivatives. Under this enlargement, if $\Phi(\bar{z}^1,\cdots,\bar{z}^p)=F(z^1,\cdots,z^p;a)\cdot\Phi(z^1,\cdots,z^p)$, we say that Φ is conformally invariant.) The invariant solutions of the system $\Phi_s=0$ are those solutions $y_k=\varphi_k(x^1,\cdots,x^m)$ which remain invariant under the transformations of G_1, that is, $\bar{y}_k=\varphi_k(\bar{x}^1,\cdots,\bar{x}^m)$. The author's principal result is that for an invariant system of equations, the invariant solutions can be expressed in terms of the solutions of another system of equations in $m-1$ independent variables, with the aid of certain invariants of the group G_1. He applies this to a single 4th order equation arising in the theory of elasticity and to a system of 2 equations which occur in the theory of fluid mechanics. *D. L. Bernstein* (Rochester, N. Y.).

Integral Equations

***Hilbert, David. Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen.** Chelsea Publishing Company, New York, N. Y., 1953. xxvi+282 pp. $4.50.

Reprint by photo-offset of the edition of 1912 [Teubner, Leipzig, 1912].

***Hellinger, E., und Toeplitz, O. Integralgleichungen und Gleichungen mit unendlichvielen Unbekannten.** Chelsea Publishing Company, New York, N. Y., 1953. 1335–1616 pp. $4.50.

Reprinted from the 1928 edition [Teubner, Leipzig].

Zaanen, A. C. An extension of Mercer's theorem on continuous kernels of positive type. Simon Stevin **29**, 113–124 (1952).

The author proves the following extension of Mercer's theorem. Let $K(x, y)$ be a non-negative definite Hermitian kernel defined in $a\leqq x\leqq b$, $a\leqq y\leqq b$ such that

$$\int_a^b |K(x,y)|^2dy<\infty\qquad(a\leqq x\leqq b),$$

$$\int_a^b |K(x',y)-K(x,y)|^2dy\to 0\quad(x'\to x).$$

Suppose also that $K(x, y)$ is continuous as a function of (x, y) at every point (x, x) on the diagonal of the square of definition. Then $K(x, y)$ is equal almost everywhere to a continuous kernel $K_c(x, y)$, and the bilinear formula

$$K_c(x,y)=\sum_{n=1}^{\infty}\lambda_n\varphi_n(x)\overline{\varphi_n(y)}$$

in terms of the eigen-values and eigen-functions of $K(x, y)$ holds in the sense of uniform convergence. A further extension of this result is stated without proof. Similar, but not overlapping, results were given by E. W. Hobson [Proc. London Math. Soc. (2) **14**, 5–30 (1915)].

F. Smithies (Cambridge, England).

Laasonen, Pentti. Eine Verallgemeinerung des symmetrischen Kernes einer Integralgleichung. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **118**, 33 pp. (1952).

The homogeneous integral equation

$$u(x)=\lambda\int K(x,y)u(y)dy$$

is considered under the following assumptions: $u(x)$ is a vector-function, $K(x, y)$ an appropriate square matrix with real elements, x and y running in a limited domain D of a number space of finite dimension. D is divided into a finite number of subdomains by means of analytical surfaces. A positive definite or positive semidefinite symmetrical matrix $S(x)$ with real elements shall exist, such that $S(x)K(x,y)=K'(y,x)S(y)\neq 0$ and $K(x,y)=K^*(x,y)S(y)$. Here K' is the transposed matrix of K, and S, K, K^* shall be continuous in each of the subdomains. The typical difficulty is that $S(x)$ is not definite, otherwise the equation could be easily transformed into another one with a symmetrical kernel-matrix.

The author attacks the problem by means of the classical iteration method of Erhard Schmidt and in a second way by the variational method of R. Courant. With respect to the scalar-product $(u,v)=\int u'(x)S(x)v(x)dx$ for two real vector-functions u and v, the results are: existence of at least one eigenvalue, reality, finite multiplicity and no finite limiting value of eigenvalues, orthogonality of eigenvector-functions of different eigenvalues and an expansion theorem for vector-functions.

It seems that a not explicitly mentioned assumption for these results is that the elements of $K(x, y)$ are bounded. The paper contains also some remarks on such kernel-matrices where $K^*(x, y)$ does not exist and where $S(x)$ is unsymmetrical. The reviewer believes that the main theorem of a paper of H. Wielandt [Math. Nachr. **4**, 308–314 (1951); these Rev. **12**, 717] could give an easy access to the same results. Wielandt's theorem refers to Hermitian operators in a complex linear space with a semidefinite scalar-product under assumptions which correspond mainly to Fredholm's theorems for nonhomogeneous integral equations.

H. Bückner (Holloman, N. M.).

Roberts, J. H. A nonconvergent iterative process. Proc. Amer. Math. Soc. 4, 640–644 (1953).

The integral equation $y(t)=\int_0^t G[y(x)][\pi(t-x)]^{-1/2}dx$ with $G(y)$ defined for $y\geqq 0$ is considered. Mann and Wolf have shown that a unique bounded solution y exists for $t\geqq 0$ if $G(y)$ is continuously defined for $y\geqq 0$, if $G(1)=0$ and if G is strictly decreasing. The solution y is strictly increasing with $y\rightarrow 1$ when $t\rightarrow\infty$. Assuming a Lipschitz condition on G, Mann and Wolf have demonstrated that the iterated sequence $y_0(t)=0$, $y_{n+1}(t)=\int_0^t G(y_n^*)[\pi(t-x)]^{-1/2}dx$, $y_n^*=\min(y_n, 1)$ converges to the solution. The author proves, that the convergence does not take place for

$$G(y)=1-y \quad (0\leqq y\leqq \tfrac{1}{2}), \quad G(y)=\tfrac{1}{2}[1-(2y-1)^{1/3}] \quad (y>\tfrac{1}{2}).$$

Another result is that convergence takes place if $G(y)$ is convex for $0\leqq y\leqq 1$, the sense of convexity being the same as for y^2. *H. Bückner* (Holloman, N. M.).

Functional Analysis

Dieudonné, J. Sur les propriétés de permanence de certains espaces vectoriels topologiques. Ann. Soc. Polon. Math. 25 (1952), 50–55 (1953).

L'auteur démontre qu'un sous-espace vectoriel H de codimension finie d'un espace vectoriel bornologique E est bornologique et que la proposition analogue est vraie pour des espaces tonnelés [cette note suit les notations et la terminologie introduites dans N. Bourbaki, Ann. Inst. Fourier Grenoble **2**, 5–16 (1951); ces Rev. **13**, 137; et J. Dieudonné et L. Schwartz, ibid. **1**, 61–101 (1950); ces Rev. **12**, 417]. Ces théorèmes sont vraies trivialement lorsque H est fermé. Au cas contraire, l'auteur introduit une nouvelle topologie dans E telle que (1) H soit fermé et (2) les topologies induites sur H par la topologie donnée et par la nouvelle topologie soient identiques. La démonstration utilise un théorème de Mackey [Trans. Amer. Math. Soc. **57**, 155–207 (1945), theorem IV-8; ces Rev. **6**, 274; **7**, 620] dont il donne une démonstration directe. *G. K. Kalisch.*

Klee, V. L., Jr. Convex sets in linear spaces. III. Duke Math. J. 20, 105–111 (1953).

[For parts I and II see same J. **18**, 443–466, 875–883 (1951); these Rev. **13**, 354, 849.] Define an open set as one whose intersection with any finite Euclidean space is open. (This is the counterpart of the popular weak topology for complexes.) This procedure does not yield a Hausdorff linear space L for $\dim L>\aleph_0$, for addition is not continuous, but for $\dim L\leqq\aleph_0$ it does and indeed determines the strongest topology for L as a locally convex linear topological space. The strongest Hausdorff linear topology is not locally convex for $\dim L\geqq 2^{\aleph_0}$. The latter part of the paper contains some remarks on generating convex Borel sets.

D. G. Bourgin (Urbana, Ill.).

Livingston, Arthur E. The space H^p, $0<p<1$, is not normable. Pacific J. Math. 3, 613–616 (1953).

This note shows that, like the L^p and l^p spaces, the space H^p is not locally convex when $0<p<1$. This, with the result of Walters [Proc. Amer. Math. Soc. **1**, 800–805 (1950); these Rev. **12**, 616] that L^{p*} is total over L^p, suggests that H^p resembles l^p more than L^p. (It would be interesting to know whether H^p is isomorphic to l^p.) *M. M. Day.*

Allen, H. S. On groups of infinite matrices. Nederl. Akad. Wetensch. Proc. Ser. A. 56 = Indagationes Math. 15, 223–230 (1953).

L'auteur étudie les groupes d'automorphismes continus d'un espace de Köthe α, qui laissent invariant globalement (resp. ponctuellement) un sous-espace β de α; il dénote un tel groupe $h(\beta)$ (resp. $g(\beta)$). Il montre par exemple que la relation $h(\lambda)\supset h(\beta)$ implique $\lambda=\beta$ si ces deux sous-espaces sont fermés, ou que $h(\beta)$ est le normalisateur de $g(\beta)$ lorsque β est fermé. Il envisage aussi plus particulièrement le cas où β admet un supplémentaire topologique λ, α ayant une topologie forte d'espace de Banach, et montre que l'on a alors $h(\beta)h(\lambda)=g(\beta)g(\lambda)$. La plupart des résultats de l'auteur s'étendent sans modification au cas d'un espace localement convexe séparé quelconque, et leur démonstration serait beaucoup plus simple si l'auteur utilisait un langage plus géométrique (transvections); mais il paraît ignorer tous les travaux faits depuis 1940 sur les espaces localement convexes. *J. Dieudonné* (Evanston, Ill.).

Charzyński, Z. Sur les transformations isométriques des espaces du type (F). Studia Math. 13, 94–121 (1953).

This paper proves that an isometry of two n-dimensional F spaces for which the origins correspond is linear. The argument is of the renorming type. *D. G. Bourgin.*

Shimoda, Isae. Notes on general analysis. II. J. Gakugei Tokushima Univ. 3, 12–15 (1953).

The paper deals with analytic functions whose arguments and values lie in complex Banach spaces. The following theorems about such functions are proved: (1) A function is a homogeneous polynomial of degree n if and only if it is defined and analytic on the whole space, and homogeneous of degree n. (2) (Generalization of Schwarz's lemma) If $f(0)=0$, if f is analytic and $\|f(x)\|\leqq M$ when $\|x\|<R$, then $R\|f(x)\|\leqq M\|x\|$ when $\|x\|<R$. Equality here can occur at a point without occurring at all points. Two more theorems are stated. These are part of a study of the boundary of the maximal open set in which a power series converges. This part of the paper is obscure to the reviewer, partly because of lack of clarity in a definition, and partly because of an apparent confusion between the failure of a power series to converge at a point, and the lack of an analytic continuation into a neighborhood of that point. *A. E. Taylor.*

Halperin, Israel. Function spaces. Canadian J. Math. 5, 273–288 (1953).

Let S be a set and γ a completely additive measure defined over a σ-ring of subsets of S. The author considers two types of generalizations of L^p spaces defined over S. If $w(P)$ is a non-negative real-valued function on S, the space $L^p{}_{(w)}$ is the space of real-valued functions on S for which $|f|_p=(\int_S|f(P)|^p w(P)d\gamma(P))^{1/p}<\infty$ with $|f|_p$ as norm. The second type of function space $M^p{}_{(w)}$ is defined by first substituting for $f(P)$, $w(P)$ their non-decreasing, left-continuous rearrangements, $f^*(x)$, $w^*(x)$ which are defined on the real interval $[0, \gamma(S)]$ as follows: $f^*(0)=\text{ess sup}_S f(P)$, $f^*(x)=\sup[k|\gamma\{P|f(P)>k\}\geqq x]$, $0<x\leqq\gamma(S)$. The space of functions $g(P)$ for which

$$[g]_q=\left(\int_0^{\gamma(S)}[g^0(x)/w^*(x)]^q w^*(x)dx\right)^{1/q}$$

is the space $M^q{}_{(w)}$ where $g^0(x)$ is the level function associated with $g^*(x)$ and defined by

$$g^0(x)=\left\{\int_{a_1}^{b_1}g^*(x)dx\Big/\int_{a_1}^{b_1}w^*(x)dx\right\}g^*(x)$$

if $[a_1, b_1]$ is a maximal interval for which

$$\left\{\int_{a_1}^{t} g^*(x)dx \Big/ \int_{a_1}^{t} w^*(x)dx\right\} \leqq \left\{\int_{a_1}^{b_1} g^*(x)dx \Big/ \int_{a_1}^{b_1} w^*(x)dx\right\},$$

$a_1 \leqq t \leqq b_1$, and $g^0(x) = g^*(x)$ for all other x in $[0, \gamma(S)]$. It is shown that $L^p{}_{(w)}$, $M^q{}_{(w)}$ are normed linear spaces for $1 \leqq p, q < \infty$ and that if $1/p + 1/q = 1$, they are conjugate to each other and hence are both reflexive. Various other properties of the two spaces are investigated, in particular, a Hölder type inequality which is the basic tool in the theorems quoted. *R. E. Fullerton* (Madison, Wis.).

Rothe, E. H. A note on the Banach spaces of Calkin and Morrey. Pacific J. Math. 3, 493–499 (1953).

It is first observed that the spaces P_α of Calkin and Morrey are uniformly convex, and hence reflexive, for $\alpha > 1$. Consequently a closed bounded convex set C in such a space P_α is compact in the weak topology. Next it is observed that a real function $I(z)$ defined on such a set C is lower semicontinuous in the weak topology at each point z_0 for which there is a linear continuous functional K satisfying $I(z) - I(z_0) \geqq K(z - z_0)$. From these remarks a theorem on the existence of a minimum for a multiple integral $\int f(t, z, p)dt_1 \cdots dt_n$ in a set C is at once deducible. However, this theorem (p. 497) has a rather restricted scope, since the third hypothesis requires that the integrand f is convex in the variables z and p together. Also the hypotheses seem to imply that $\alpha \geqq 2$. *L. M. Graves* (Chicago, Ill.).

Hille, Einar. A note on Cauchy's problem. Ann. Soc. Polon. Math. 25 (1952), 56–68 (1953).

Let U be a closed linear operator whose domain $D(U)$ is dense in a complex B-space Y, and consider the equation

(1) $\quad y'(t) = \text{strong lim } h^{-1}(y(t+h) - y(t)) = U(y(t)), \; t > 0.$

The author discusses the Cauchy problem (C.P.) for (1): Given $y_0 \in Y$, find a function $y(t)$ on $(0, \infty)$ to Y such that (i) $y(t)$ and $y'(t)$ are in Y for $t > 0$, (ii) $y(t)$ satisfies (1) for $t > 0$, (iii) strong $\lim_{t \downarrow 0} y(t) = y_0$. By virtue of the semigroup theory [the author, Functional analysis and semi-groups, Amer. Math. Soc. Colloq. Publ., v. 31, New York, 1948; these Rev. 9, 594; the reviewer, J. Math. Soc. Japan 1, 15–21 (1948); these Rev. 10, 462] the uniqueness and the solvability theorems are obtained. Uniqueness: Let the resolvent $R(\lambda, U)$ exist for $\lambda > \lambda_0 \geqq 0$ such that $\lim_{\lambda \to \infty} \lambda \|R(\lambda, U)\| < \infty$. Then for any $y_0 \in Y$, the C.P. has at most one solution satisfying $\lim \sup_{t \to \infty} t^{-1} \log \|y(t)\| < \infty$. Solvability: If U is the infinitesimal generator of a semi-group $T(t)$, $0 \leqq t < \infty$, satisfying strong $\lim_{t \downarrow 0} T(t)y = y$ and strong $\lim_{t \to \infty} t^{-1} \log \|T(t)\| = \omega < \infty$, then the C.P. is (uniquely) solvable for $y_0 \in D(U)$ with

$$\|y(t)\| \leqq M\|y_0\| \exp(\omega t)$$

where M is a constant. These results are illustrated by concrete examples relating to the simple equations

$$\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = 0, \quad \frac{\partial^2 y}{\partial t^2} + \frac{\partial^2 y}{\partial x^2} = 0 \quad \text{and} \quad \frac{\partial y}{\partial t} - \frac{\partial^2 y}{\partial x^2} = 0.$$

K. Yosida (Osaka).

Mertvecova, M. A. Analogue of the process of tangent hyperbolas for general functional equations. Doklady Akad. Nauk SSSR (N.S.) 88, 611–614 (1953). (Russian)

The process of the title is a cubically converging iteration for solving an equation $f(x) = 0$ in the complex field, as discussed by Salehov [same Doklady (N.S.) 82, 525–528 (1952); these Rev. 14, 91]. Following the work of Kantorovich and others for Newton's method, the author extends the process to solve (*) $P(x) = 0$, where P carries a normed space X into a like space Y, and has a third Fréchet differential. The formula is $x_{n+1} = x_n - Q_n \Gamma_n P(x_n)$, where $\Gamma_n = [P'(x_n)]^{-1}$ and $Q_n = [I - 2^{-1}\Gamma_n P''(x_n)\Gamma_n P(x_n)]^{-1}$. In two theorems the author gives conditions for (*) to have a unique solution x^* in a region of X, and estimates the size of $\|x_n - x^*\|$, which vanishes like $2^{-n}C^{3^n}$, $0 < C < 1$. The process is applied to a class of nonlinear integral equations, and one numerical example is cited. *G. E. Forsythe.*

Rubin, H., and Stone, M. H. Postulates for generalizations of Hilbert space. Proc. Amer. Math. Soc. 4, 611–616 (1953).

Let X be a linear space with elements $x, y, \cdots$ and scalar multipliers all real, or all complex, or all quaternionic. Let q be a function on X to the set of non-negative real numbers, satisfying the (Jordan-von Neumann type) condition (*) $q(x+y) + q(x-y) = 2q(x) + 2q(y)$, and put

$$p(x, y) = \tfrac{1}{4}[q(x+y) - q(x-y)].$$

If it be assumed, moreover, that for α real, $q(\alpha x)$ is a bounded function of α in the neighborhood of the origin, and in the complex and quaternionic cases that $q(x) = q(ix)$, $q(x) = q(ix) = q(jx) = q(kx)$, respectively, it is proved that (1) in the real case, $(x, y) = p(x, y)$ is symmetric, positive semi-definite, linear in both arguments, and consequently serves as an inner product of the elements x, y; (2) in the complex case, $(x, y) = p(x, y) + ip(x, iy)$ has the usual properties of an inner product in complex spaces, while (3) in the quaternionic case, analogous properties are enjoyed by $(x, y) = p(x, y) + ip(x, iy) + jp(x, jy) + kp(x, ky)$. In all cases, $x, y \in X$ implies $G(x, y) \geqq 0$, where $G(x, y)$ denotes the Gram determinant of the two elements. In (2) and (3), as in (1), the inner product is positive semi-definite merely. In all three cases, $(x, x)^{\frac{1}{2}}$ is a semi-norm, possessing all properties of a norm except for vanishing only when $x = \theta$, the zero element of X. If it be assumed that $q(x) = 0$ implies $x = \theta$, then all inner products are positive definite, and the semi-norm is a norm. Thus condition (*), supported by very mild additional assumptions, suffices to define a norm as well as an inner product in a linear space X, effecting thereby an economy over the usual procedure.

L. M. Blumenthal (Columbia, Mo.).

Dixmier, J. Sur une inégalité de E. Heinz. Math. Ann. 126, 75–78 (1953).

This paper improves an inequality for Hilbert space operators obtained by Heinz in a recent paper on perturbation theory [Math. Ann. 123, 415–438 (1951); these Rev. 13, 471]. The same result has been obtained by Kato [ibid. 125, 208–212 (1952); these Rev. 14, 766] by a different method. The two methods, however, do seem to have non-equivalent generalizations, that of Dixmier to inequalities for multilinear forms and that of Kato to inequalities for analytic functions of the original operators.

F. H. Brownell (Princeton, N. J.).

Schäffer, Juan J. On some problems concerning operators in Hilbert space. Anais Acad. Brasil. Ci. 25, 87–90 (1953).

The proof of Wielandt [Math. Ann. 121, 21 (1949); these Rev. 11, 38] that the unit element of a Banach algebra cannot be a commutator immediately yields

1) if $AB-BA=C$ and C commutes with either A or B then a sequence of elements D_n exists such that $\|D_n\|=1$ $\lim_{n\to\infty}\|D_nC\|=\lim_{n\to\infty}\|CD_n\|=0$; 2) if $AB-BA=A^*$, then $\lim_{n\to\infty}\|A^n\|^{1/n}=0$; and 3) if $AB-BA=A$, then A is a proper nilpotent. It is shown that, for some bounded operators A, B in infinite-dimensional Hilbert space, $\|AB-BA-I\|<1$. It is announced that if a set D surrounds the origin not only the corresponding analytic position operator [Halmos, Lumer, and Schäffer, Proc. Amer. Math. Soc. **4**, 142–149 (1953); these Rev. **14**, 767] has no root of any order (except one) but also no sufficiently near (in terms of norm) operator has a root. *S. Sherman.*

Tillmann, Heinz Günther. Spektraltheoriefreie Gleichungstheorie im Hilbertschen Raum. Math. Z. **58**, 85–97 (1953).

The problem here considered is the following: If y is a vector in Hilbert space and A is a linear operator, to find all vectors x such that $Ax=y$. In matrix terminology this is formulated: $\sum_k a_{ik}x_k=y_i$. Thus the question is to find suitable coordinate systems so that the matrix $\{a_{ik}\}$ will be as simple as possible. This problem has received a solution for general bounded operators and also for the special case in which A is self-adjoint and bounded [Schmeidler, J. Reine Angew. Math. **163**, 135–140 (1930)]. The present author extends these results to closed operators. The fact that spectral theory is not invoked in the proofs is stressed. To these ends the author uses various results of Dixmier and Köthe. There is also a discussion of the affinity and equivalence of operators and of Julia operators. *E. R. Lorch.*

***Tillmann, Heinz-Günther. Gleichungstheorie im Hilbertschen Raum.** Dissertationen der mathematisch-naturwissenschaftlichen. Fakultät der westfälischen Landesuniversität Münster in Referaten, Heft 1, pp. 18–20. Aschendorffsche Verlagsbuchhandlung, Münster, 1952.

Šilov, G. E. Homogeneous rings of functions. Amer. Math. Soc. Translation no. **92**, 65 pp. (1953).

Translated from Uspehi Matem. Nauk (N.S.) **6**, no. 1(41), 91–137 (1951); these Rev. **13**, 139.

Calculus of Variations

Have

***Kimball, W. S. Calculus of variations by parallel displacement.** Butterworths Scientific Publications, London, 1952. viii+543 pp. $7.50.

This book is not written in the usual language of modern analysis, which makes it difficult to give any detailed criticism. It is not recommended for the beginner in the field.
L. M. Graves (Chicago, Ill.).

Baiada, Emilio. Su una classe particolare di problemi di calcolo delle variazioni. Ann. Scuola Norm. Super. Pisa (3) **6** (1952), 173–186 (1953).

This paper gives an existence theorem for a minimum problem in the plane in nonparametric form, in terms of a combination of conditions imposed on it and conditions imposed on the associated problem in parametric form. It is difficult to see what the precise force of the conditions is. The author makes no reference to the penetrating investigation by McShane based on the same idea [same Ann. (2) **3**, 183–211, 239–241, 287–315 (1934)]. *L. M. Graves.*

Bertolini, Fernando. A new proof of the existence of the minimum for a classical integral. Compositio Math. **11**, 37–43 (1953).

A proof is given for the existence of a minimum for integrals of the form $\int y^n(x'^2+y'^2)^{1/2}dt$, using a method of Picone.
L. M. Graves (Chicago, Ill.).

Cinquini, Silvio. Sopra gli integrali doppi del calcolo delle variazioni dipendenti dalle derivate del secondo ordine. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **15**(**84**), 327–336 (1951).

The author extends his work on the existence of a minimum for a double integral $\iint F(x,y,z,p,q,r,s,t)dxdy$ to cases in which the integrand is not bounded below [see Ann. Scuola Norm. Super. Pisa (2) **10**, 215–248 (1941); these Rev. **8**, 388]. *L. M. Graves* (Chicago, Ill.).

Darbo, Gabriele. Sulle condizioni sufficienti per la continuità di un integrale. Rend. Sem. Mat. Univ. Padova **22**, 134–142 (1953).

This note gives sufficient conditions for uniform continuity of the functional $I(y)=\int_{x_1}^{x_2}Q(x,y)y'dx$ in the class of absolutely continuous functions $y(x)$ with graphs in a fixed rectangle. These conditions are that $Q(x,y)$ is bounded and measurable and satisfies

$$|Q(x',y)-Q(x'',y)|\leqq|F(x')-F(x'')|\phi(y),$$

where F is monotonic and ϕ is integrable.
L. M. Graves (Chicago, Ill.).

Föllinger, Otto. Ebene Variationsprobleme mit freien Endpunkten. Math. Z. **58**, 98–112 (1953).

This paper treats the problem of minimizing an integral $\int F(x,y,x',y')dt$ in the plane in the case when both end-points are entirely free and also in the case when one end point is free while the other is restricted to a curve. The case when only one end point is free was treated earlier by Rasmadsé [Math. Ann. **75**, 380–401 (1914)]. By means of the "Risspunktkurve" the conditions derived are related to those derived by Bliss [ibid. **58**, 70–80 (1904)] for the case of two variable end-points [later treated in another manner by Dresden, Trans. Amer. Math. Soc. **9**, 467–486 (1908)]. It seems likely that use of the formulas of Dresden would make possible a much shorter proof of the results of this paper. There seems to be an error in the last paragraph on p. 99, where it is stated that the three functions $x=\xi(a)$, $y=\eta(a)$, $\Theta=\vartheta(a)$ satisfy the Euler-Lagrange equations.
L. M. Graves (Chicago, Ill.).

Bertolini, Fernando. Proprietà di minimo delle curve di Ribaucour. Ann. Mat. Pura Appl. (4) **34**, 161–194 (1953).

This paper is concerned with a detailed study of the fixed end point problem in the plane variational problem involving the integral $\int y^{1/n}ds$. The author was evidently unaware of a paper by G. Mammana [same Ann. (4) **10**, 1–31 (1932)] devoted to the same problem for the case $n>0$.
W. T. Reid (Evanston, Ill.).

Föllinger, Otto. Diskontinuierliche Lösungen mit Spitzen in der Variationsrechnung. Arch. Math. **4**, 121–132 (1953).

It is known that in general a curve having a cusp cannot furnish a strong relative minimum to a calculus of variations integral. This paper presents sufficient conditions for a curve with a cusp to furnish a weak relative minimum, in

a sense which the author defines. The reviewer was not able to follow some of the proofs. *L. M. Graves.*

Èl'sgol'c, L. È. Variational problems with retarded argument. Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk **1952**, no. 10, 57–62 (1952). (Russian)

The author calls attention to a class of variational problems with retarded arguments. As an example he derives differential-difference equations analogous to the Euler equations for the problem given by

$$V(x(t))=\int_{t_0}^{t_1}F(t, x(t-\tau_1(t)), \cdots, x(t-\tau_n(t)), \cdots, \dot{x}(t-\tau_1(t)), \cdots, \dot{x}(t-\tau_n(t)))dt,$$

where $x(t_0-\tau_i(t_0))$ and $x(t_1-\tau_i(t_1))$ are given, the lag functions $\tau_i(t)$ are nondecreasing continuously differentiable functions with $\tau_i'(t)<1$ $(i=1, 2, \cdots, n)$, and admissible x's are smooth functions defined on $(t_0-\max_i \tau_i(t_0))$. Proceeding, he proves that if $x(t)$ minimizes V then the quantities

$$F_{\dot{x}_i\dot{x}_j}(t, x_1, \cdots, x_n, \dot{x}_1, \cdots, \dot{x}_n),$$

where $x_i=x(t-\tau_i(t))$ and $\dot{x}_i=\dot{x}(t-\tau_i(t))$, are the coefficients of a positive semi-definite quadratic form.

J. M. Danskin (Washington, D. C.).

Fet, A. I. Variational problems on closed manifolds. Amer. Math. Soc. Translation no. **90**, 61 pp. (1953).

Translated from Mat. Sbornik N.S. **30**(72), 271–316 (1952); these Rev. **13**, 955.

Kondrašov, V. I. On the theory of nonlinear and linear problems on characteristic values. Doklady Akad. Nauk SSSR (N.S.) **90**, 129–132 (1953). (Russian)

Let W be the class of real functions $u(x)$ $(x=x_1, \cdots, x_n)$, defined in a domain D bounded by manifolds S_{n-s} of dimensions $n-s=1, \cdots, n-1$, which possess partial derivatives $y_1, \cdots, y$ of order m whose pth powers are integrable in D. Let $F(u)$ be a polynomial of degree p in the $|y|$ whose coefficients are functions of x and let $G(u)$ be a similar expression in terms of derivatives of orders $\leqq m-\lambda$, the degree being q, subject to $1\leqq\lambda\leqq m$, $1<q<np(n-p\lambda)^{-1}$ and to the restriction of admissibility, i.e., integrability of $F(u)$ and $G(u)$ in D for each $u \in W$.

The author considers the problem of minimum of $\int_D F(u)dx$ for functions $u \in W$ subject to $\int_D G(u)dx=1$ and to the partial derivatives of orders $<m-(s/p)$ vanishing in the mean on S_{n-s}, and announces a number of propositions, which generalize the quadratic case $p=q=2$ and which connect the problem to eigen-values and eigen-functions. The absence of proofs and the condensed form of the definitions, together with a probable misprint in equation (1), detract somewhat from the clarity of the text.

L. C. Young (Madison, Wis.).

Theory of Probability

***Kawada, Yukiyoshi. Kakuritsuron. [The theory of probability].** Rev. ed. Kyôritsusha, Tokyo, 1952. 2+2+377+4+4+2 pp. 550 yen.

This is a text-book for advanced graduate students. The author tries to give a comprehensive exposition of the modern probability theory by combining the analytical method of H. Cramér [Random variables and probability distributions, Cambridge, 1937] and the probabilistic method of P. Lévy [Théorie de l'addition des variables aléatoires, Gauthier-Villars, Paris, 1937].

The book consists of seven chapters. In the first chapter probability is introduced as a theory of normalized measure algebras, i.e., σ-Boolean algebras with finitely or countably additive normalized measures defined on them. The classical law of large numbers of Bernoulli type and the notion of collectives of R. von Mises are discussed. The second chapter is on Markoff processes with a finite number of possible states. Decomposition of the total space into dissipative, ergodic and cyclic parts is discussed. Poincaré's problem of card shuffling is treated as a problem of left-invariant Markoff processes on a finite group of permutations.

In chapter 3, probability is introduced as a theory of normalized measure spaces, i.e., σ-fields of subsets of a set with countably additive normalized measures defined on them. The following notions are discussed: random variables, distribution functions and their characteristic functions, independence and conditional probabilities. Measures on infinite product spaces are discussed following Kolmogoroff [Grundbegriffe der Wahrscheinlichkeitsrechnung, Springer, Berlin, 1933].

Chapters 4 and 5 are on convergence of probability distributions and random variables. The central limit theorem and its generalizations by P. Lévy [loc. cit. Chapter 5] are discussed in chapter 4. The notion of dispersion introduced by P. Lévy [loc. cit.] plays a fundamental role here. Relations between various types of convergence (almost everywhere convergence, mean convergence, convergence in probability and convergence in law) are discussed in Chapter 5. Equivalence of these types of convergence in the case of sums of independent random variables is proved by using the maximal concentration function of P. Lévy [loc. cit.]. The three-series theorem of A. Khintchine and A. Kolmogoroff [Mat. Sbornik **32**, 668–677 (1925)] is proved. The strong law of large numbers of A. Kolmogoroff [loc. cit.] and the law of the iterated logarithm of A. Khintchine [Asymptotische Gesetze der Wahrscheinlichkeitsrechnung, Springer, Berlin, 1933] are discussed.

Chapter 6 deals with ergodic theorems. The individual ergodic theorem of G. D. Birkhoff is proved following A. Kolmogoroff [Mat. Sbornik N.S. **2**(44), 367–368 (1937)] and the mean ergodic theorem of J. von Neumann is proved following G. Birkhoff [Duke Math. J. **5**, 19–20 (1939)]. Ergodicity and, in particular, weak and strong mixing properties of measure preserving transformations are discussed. Various theorems due to E. Hopf and the author concerning weakly mixing transformations on a direct product measure space are proved. The random ergodic theorem of S. M. Ulam and J. von Neumann [Bull. Amer. Math. Soc. **51**, 660 (1945)] is discussed at the end of this chapter.

Chapter 7 is on stationary processes. Ergodicity and mixing properties of the shift transformation of stationary processes are discussed. Various results of ergodic type concerning stationary Markoff processes are obtained following J. L. Doob [Trans. Amer. Math. Soc. **44**, 87–150 (1938)]. Further, stationary time series are discussed from the standpoint of the theory of collectives of R. von Mises. This chapter is concluded with a brief discussion of stationary Gauss processes.

The revised edition has an appendix [not in the 1948 edition] of 38 pages by K. Takano on limit theorems for sums of dependent random variables and on infinitely divisible laws. The exposition follows an article by M. Loève

[Univ. Calif. Publ. Statist. **1**, 53–87 (1950); these Rev. **12**, 425]. *S. Kakutani* (New Haven, Conn.).

Curtiss, J. H. Elements of a mathematical theory of probability. Math. Mag. **26**, 233–254 (1953).

The discussion proceeds on two levels: the mathematical theory, and the conceptual and experimental background. The mathematical theory is simply the familiar measure-theoretical formulation of probability according to Kolmogorov and others. The treatment of the conceptual side is mainly based on the use of frequencies as suggesting the method of measuring probabilities in actual cases, as well as the motivation of the mathematical postulates. The author is aware of the impossibility of defining "randomness" of sequences, as attempted by von Mises. On the other hand, the notion that the trials in the set must be independent and performed under sufficiently similar conditions is stated in vague intuitive terms, apparently without the realization that an intuitive conception of probability is entering here. Furthermore, the usual difficulty of the stability of frequency ratios is answered by asserting that using the value of a frequency is making a hypothesis as to the value of a probability, to be progressively amended by further observations. This leaves the import of the word "amended" obscure. Moreover, it seems not to be recognized that, at any stage, a hypothesis is accepted as practically useful only if it has some sort of presumed validity or cogency—"probability", as one would say in usual parlance. Finally, in mentioning more flexible schemes in which all events do not necessarily have probabilities, the only reference is to R. Fortet [Calcul des probabilités, Centre National de la Recherche Scientifique, Paris, 1950; these Rev. **12**, 423] whereas the reviewer has published such a theory earlier [Ann. of Math. (2) **41**, 269–292 (1940); **42**, 169–187 (1941); these Rev. **1**, 245; **2**, 227]. Nor does the author note the possibility of a development based on probability as order, more basic than probability as a number, as contained in treatments by Vietoris and the reviewer.

B. O. Koopman (New York, N. Y.).

★de Finetti, Bruno. Sulla nozione di "dispersione" per distribuzioni a più dimensioni. Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 587–596. Casa Editrice Perrella, Roma, 1953.

Let F be a probability measure in r dimensions, and let $q(m)=\sup F(E)$ for E in a specified class of convex sets, and E of Lebesgue measure m. It is supposed that the class of sets contains every translation of each of its sets. The author discusses the character of a maximizing set for specified m and various classes of convex sets.

J. L. Doob (Urbana, Ill.).

Hammersley, J. M. Corrigenda to the paper, Tauberian theory for the asymptotic forms of statistical frequency functions. Proc. Cambridge Philos. Soc. **49**, 735 (1953).

See same Proc. **48**, 592–599 (1952); these Rev. **14**, 387.

Andreoli, G. Saggio matematico di leggi evolutive di collettività (non-determinismo, causalismo, finalismo). Ricerca, Napoli **2**, no. 3–4, 3–9 (1951); **3**, no. 1, 3–7, no. 2, 3–11, no. 3–4, 17–24 (1952); **4**, no. 1–2, 11–39 (1953).

The author discusses in an elementary manner the development of a population which falls into different classes. It is assumed that the changes of the population are due to one of the following causes: (a) Transition of elements from one class to another (b) multiplication of elements (c) combination of causes (a) and (b). No use is made of the theory of Markoff chains or of the theory of branching processes. *E. Lukacs* (Washington, D. C.).

Tecedor, Sixto Camara. Transformations of the laws of probability. Euclides, Madrid **10**, 390–396, 433–442 (1950); **11**, 5–11, 70–76, 170–176, 251–254 (1951). (Spanish)

Ostrowski, A. M. Two explicit formulae for the distribution function of the sums of n uniformly distributed independent variables. Arch. Math. **3**, 451–459 (1952).

Let $x_1, x_2, \cdots, x_n$ be independent random variables, such that x_i is uniformly distributed between the limits $\pm\alpha_i$. The author proves a formula due to Olds [Ann. Math. Statistics **23**, 282–285 (1952); these Rev. **14**, 64] and Rufener [Mitt. Verein. Schweiz. Versich.-Math. **52**, 97–120 (1952); these Rev. **13**, 956] for the distribution function of the sum $x_1+\cdots+x_n$, and also a related formula involving certain conditional probabilities. *H. Cramér.*

Linnik, Yu. V. On some identically distributed statistics. Doklady Akad. Nauk SSSR (N.S.) **89**, 9–11 (1953). (Russian)

This is a continuation of a previous paper [same Doklady (N.S.) **83**, 353–355 (1952); these Rev. **14**, 60] and we refer to the previous review. A simple sufficient condition that (B) implies (A) is given. A theorem is given where the normal law in (A) is replaced by a convolution of symmetrical stable laws, followed by two theorems on characteristic functions, of independent interest. The second asserts that if a ch. f. is of the form $e^{P(u)}$ in $-\delta\leqq u\leqq\delta$ where P is a polynomial, then it is so for all real u. The maximum of a sequence of independent random variables is the "dual" of their sum, in the sense that they correspond to the multiplication of d. f.'s and ch. f.'s resp. Two theorems of the dual type are stated. A more detailed review must await publication of proofs of these announcements. *K. L. Chung.*

Please see the corresponding note on p. 1139.

★Kunisawa, Kiyonori. Kakuritsuron ni okeru kyokugenteiri. [Limit theorems in probability theory.] Chûbunkan, Tokyo, 1949. 1+2+152 pp.

In this book the author expounds the theory started in his thesis [Ann. Inst. Statist. Math., Tokyo **1**, 1–77 (1949); also, Proc. Imp. Acad. Tokyo **20**, 627–630 (1944); these Rev. **11**, 255; **7**, 312]. The main purpose of this book is to give a unified treatment of the modern probability theory by using the analytical method of mean concentration functions. The book consists of 10 chapters. After two short introductory chapters, the mean concentration function $\Psi_F(h)$ of a distribution function $F(x)$ is defined in chapter 3 by the formula:

$$\Psi_F(h)=h\int_0^\infty e^{-ht}|f(t)|^2dt = \int_{-\infty}^\infty \frac{h^2}{h^2+x^2}d\tilde{F}(x), \quad (h>0) \tag{1}$$

where $\tilde{F}(x)=F(x)*\{1-F(-x)\}$ is the symmetrization of $F(x)$ and $f(t)$ is the characteristic function of $F(x)$. Further,

$$\Phi_F(h)=h\int_0^\infty e^{-ht}\Re f(t)dt, \quad (h>0) \tag{2}$$

is called the typical function of $F(x)$, where $\Re f(t)$ denotes the real part of $f(t)$. The author proves several fundamental inequalities concerning these functions in chapter 3. Also,

relations of these functions with the maximal concentration function of P. Lévy [Théorie de l'addition des variables aléatoires, Gauthier-Villars, Paris, 1937] and the mean concentration function of T. Kawata [Duke Math. J. **8**, 666–677 (1941); these Rev. **3**, 168] are discussed.

Chapter 4 deals with the convergence properties of series with independent terms. Let $\{X_n | n=1, 2, \cdots\}$ be a sequence of independent real-valued random variables and put $\Psi_{n,N}(h)=\sum_{k=n+1}^{N}(1-\Psi_{F_k}(h))$, where $F_k(x)$ is the distribution function of X_k. Then $\Psi_n(h)=\lim_{N\to\infty}\Psi_{n,N}(h)$ and $\Psi(h)=\lim_{n\to\infty}\Psi_n(h)$ exist for all $h>0$. It is shown that $\Psi(h)\equiv 0$ if there exists a sequence of real numbers $\{a_k | k=1, 2, \cdots\}$ such that $\sum_{k=1}^{\infty}(X_k-a_k)$ is convergent in probability, and $\Psi(h)\equiv\infty$ otherwise. From this follows the three-series theorem of Khintchine and Kolmogoroff [Mat. Sbornik **32**, 668–677 (1925)]. The equivalence of various types of convergence for the series with independent terms also follows from this.

The weak law of large numbers is discussed in chapter 5. An infinite triangular array of random variables

$$\mathfrak{X}=\{X_{n,k} | k=1, \cdots n, n=1, 2, \cdots\},$$

each row forming an independent system, is said to obey the weak law of large numbers if there exist a sequence of positive numbers $\{A_n | n=1, 2, \cdots\}$ and a sequence of real numbers $\{B_n | n=1, 2, \cdots\}$ such that

$$\lim_{n\to\infty}\{(X_{n,1}+\cdots+X_{n,n})/A_n-B_n\}=0$$

in probability. It is shown that $\mathfrak{X}$ obeys the weak law of large numbers if and only if there exists a sequence of positive numbers $\{A_n | n=1, 2, \cdots\}$ such that

$$\lim_{n\to\infty}\sum_{k=1}^{n}(1-\Psi_{F_{n,k}}(A_n))=0,$$

where $F_{n,k}$ is the distribution function of $X_{n,k}$. From this follows a result of Feller [Acta Litt. Sci. Szeged **8**, 191–201 (1937)] on the weak law of large numbers.

Chapter 6 is devoted to the discussion of Gaussian laws. Theorems of Cramér [Math. Z. **41**, 405–414 (1936)] and Kac [Amer. J. Math. **61**, 726–728 (1939); these Rev. **1**, 62] are proved. The central limit theorem is treated as a special case of a general theorem of Gnedenko [C. R. (Doklady) Acad. Sci. URSS (N.S.) **22**, 60–63 (1939)] and the author [thesis, loc. cit.] concerning the convergence to an infinitely divisible law. From this follow the theorems of P. Lévy [loc. cit.] and Feller [Math. Z. **40**, 521–559 (1935)]. This chapter is concluded with a discussion of the domain of attraction for Gaussian distributions.

In chapter 7 the uniform diminution theorems of the author [thesis, loc. cit.] are proved both for the mean concentration function $\Psi_F(h)$ and the typical function $\Phi_F(h)$. From these follow the results of W. Doeblin and P. Lévy [C. R. Acad. Sci. Paris **202**, 2027–2029 (1936)] and Doeblin [Bull. Sci. Math. (2) **63**, 23–32, 35–64 (1939)].

Infinitely divisible laws are discussed in chapter 8. A theorem of Khintchine [Mat. Sbornik N.S. **2**(**44**), 79–119 (1937)] concerning the equivalence of various definitions of infinitely divisible laws is proved. P. Lévy's general form for the characteristic function of an infinitely divisible law is obtained following Khintchine [Bull. Univ. d'Etat Moscou, Ser. Internat., Sect. A. **1**, 1–5 (1937)]. Further, results of Khintchine [Mat. Sbornik, loc. cit.] concerning partial limit laws and the results of B. Gnedenko [loc. cit.] are obtained by using the method of mean concentration functions. This chapter is concluded with a discussion of decomposition into factors of a probability law.

The power set of a probability law is discussed in chapter 9. The notion of class convergence of Khintchine [Izvestiya Naučno-Issled. Inst. Mat. Meh. Tomsk Gos. Univ. **1**, 258–262 (1937)] is introduced. Quasi-stable laws and stable laws are discussed. The results of Doeblin [Studia Math. **9**, 71–96 (1940); these Rev. **3**, 168] and Gnedenko [C. R. (Doklady) Acad. Sci. URSS (N.S.) **24**, 640–642 (1939)] are obtained by using the method of the author [Proc. Phys.-Math. Soc. Japan (3) **24**, 681–695 (1942); these Rev. **7**, 311].

The strong law of large numbers and the law of the iterated logarithm are discussed in chapter 10. A sequence $\mathfrak{X}=\{X_k | k=1, 2, \cdots\}$ of independent random variables is said to obey the strong law of large numbers if

$$\lim_{n\to\infty} n^{-1}\sum_{k=1}^{n}(X_k-a_k)=0 \quad\text{and}\quad \lim_{n\to\infty} X_n/n=0$$

with probability 1, where $a_k=\int_{-k}^{k}xdF_k(x)$ and F_k is the distribution function of X_k. It is proved that $\mathfrak{X}$ obeys the strong law of large numbers if and only if $\sum_{k=0}^{\infty}\{1-\Phi_{F_k}(k)\}<\infty$. From this follows the well known result of Kolmogoroff [Grundbegriffe der Wahrscheinlichkeitsrechnung, Springer, Berlin, 1933]. A problem of Khintchine [Giorn. Ist. Ital. Attuari **7**, 365–377 (1936)] of finding a condition for the existence of a sequence $\{A_n | n=1, 2, \cdots\}$ such that $\lim_{n\to\infty}(X_1+\cdots+X_n)/nA_n=1$ with probability 1 is discussed. The law of the iterated logarithm is proved in the form given by Kolmogoroff [Math. Ann. **101**, 126–135 (1929)]. Finally, a generalization of the results of P. Lévy [Studia Math. **3**, 119–155 (1931)] and Marcinkiewicz [Bull. Sém. Math. Univ. Wilno **2**, 22–34 (1939); these Rev. **1**, 21] is proved. In all these discussions the arguments are very much simplified by the use of the mean concentration function. *S. Kakutani* (New Haven, Conn.).

Lipschutz, Miriam. On strong laws for certain types of events connected with sums of independent random variables. Ann. of Math. (2) **57**, 318–330 (1953).

It is shown that the conditions used by Chung and Erdös ~~(unpublished paper)~~ in connection with a new method for proving strong limit theorems are satisfied for certain types of events related to sequences of sums $S_1, S_2, \cdots$ of independent identically distributed random variables. This result is then applied to obtain strong laws for the number of positive terms in the sequence $S_1, S_2, \cdots, S_n$, and for the upper bound of the number of zeros in this sequence in the case of coin-tossings. The detailed results are too complicated to be given here. *H. Cramér* (Berkeley, Calif.).

[Trans. Amer. Math. Soc. 72, 179–186 (1952); these Rev. 13, 567]

***Gasapina, Umberto. Un teorema limite del calcolo delle probabilità.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 599–609. Casa Editrice Perrella, Roma, 1953.

Let $s_n=\sum_1^n x_j$, where the summands are mutually independent, with zero means and unit variance, and are subject to the central limit theorem. The author proves that, if $\sum_1^n S_j^{2a}/n^{a+1}$ has a limiting distribution, the limiting distribution does not depend on the distributions of the x_j's. He proves this under the assumption that the x_j's have uniformly bounded moments of order $2(2a-1)$. He is evidently unaware of the work of Donsker [Mem. Amer. Math. Soc. no. **6** (1951); these Rev. **12**, 723] who proved the existence of a limit distribution, which he identified, for the case of a common distribution function of the x_j's having

only a finite second moment. This type of theorem goes back to Erdös and Kac [Bull. Amer. Math. Soc. 52, 292–302 (1946); these Rev. 7, 459] who treated the case $a=1$, among similar theorems. *J. L. Doob* (Urbana, Ill.).

Fortet, Robert, et Mourier, Edith. Lois des grands nombres pour des éléments aléatoires prenant leurs valeurs dans un espace de Banach. C. R. Acad. Sci. Paris 237, 18–20 (1953).

The authors continue their work on the law of large numbers for random variables taking on values in a separable Banach space X. The space X is of type P if it satisfies certain conditions, which, for example, are satisfied if X is L_a for $a\geqq 2$. If X is of type P, there is a constant A, depending only on the space, such that, if $x_1, x_2, \cdots$ are mutually independent random variables with $E\{\|x_j\|^2\}\leqq M$, it follows that $E\{\|\sum_1^n x_j/n\|^2\}\leqq AM/n$, and the averages converge strongly to the zero element, with probability 1. This corrects an assertion in a previous paper [same C. R. 234, 699–700 (1952); these Rev. 14, 387]. In the Gaussian case, theorems of a previous note [Mourier, ibid. 236, 575–576 (1953); these Rev. 14, 662] which assumed that X was Hilbert space are now said to be true if X is of type P.

J. L. Doob (Urbana, Ill.).

Chung, K. L., and Kac, M. Corrections to the paper "Remarks on fluctuations of sums of independent random variables". Proc. Amer. Math. Soc. 4, 560–563 (1953).

The limiting distribution of the number of changes of sign in a sequence of sums of independent random variables with a common symmetric stable distribution was given incorrectly, for exponents $\leqq 1$, in Mem. Amer. Math. Soc. no. 6 (1951); these Rev. 12, 722. *J. L. Doob.*

Lévy, Paul. Processus markoviens et stationnaires du cinquième type (infinité dénombrable d'états possibles, paramètre continu). C. R. Acad. Sci. Paris 236, 1630–1632 (1953).

Elaboration of previous work [Ann. Sci. Ecole Norm. Sup. (3) 68, 327–381 (1951); 69, 203–212 (1952); these Rev. 13, 959; 14, 663]. Let F be a set of states and let τ be the time the system spent in them. With τ as the new time parameter (namely, discarding on the t-axis all points corresponding to states outside F) a derived process is obtained which is Markovian with stationary transition. Take F to be the set of all instantaneous (inst.) states and let the derived process be P_1. For P_1 an infinity of these states are no longer inst. (since there must be an infinity of non-inst. states). Let the set (possibly empty) of the remaining inst. states be F_2 and derive a process P_2 from P_1 in the same way. Thus we obtain a possibly transfinite sequence of processes. If ρ is the smallest ordinal number (of finite or denumerable type) for which P_ρ has no inst. states, then P is of order ρ. Similarly we may define the order of a state. Previously the author has considered the case $\rho=1$. As an application, it is shown that if $\bar{E}_k$ is the closure of the t set on which the state A_k is realized, then at most a denumerable number of t belongs to more than one $\bar{E}_k$. In paragraph 3 the word "stationnaire" must mean "instantaneous" as defined in the cited papers. This is rather unfortunate since "stationary" resembles "stable" which was defined to be non-inst.

K. L. Chung (Syracuse, N. Y.).

Lévy, Paul. Rectification au texte d'une note antérieure. C. R. Acad. Sci. Paris 237, 964 (1953).

The author replaces the word "stationnaire" by "instantané" in the paper reviewed above (cf. the last two sentences of that review).

Kendall, David G. Stochastic processes occurring in the theory of queues and their analysis by the method of the imbedded Markov chain. Ann. Math. Statistics 24, 338–354 (1953).

The author's analysis of queues by the method of an imbedded Markov chain, broached in a preceding paper [J. Roy. Statist. Soc. Ser. B. 13, 151–173 (1951); these Rev. 13, 957], is clarified and extended to a system of s servers, each with an exponential distribution of service times, with service in order of arrival and inter-arrival times independent and each with the same distribution, which is left general. As usual, it is tacitly assumed that arrivals forced to wait, wait as long as is necessary. The Markov chain is specified by the epochs of arrival and the system by the number being served or waiting. The transition matrix for the chain is completely determined for any s and illustrated for $s=4$; and it is shown that the chain is ergodic if the relative traffic intensity (the occupancy, in telephone parlance) is less than unity. Also it is shown that in equilibrium conditions the size of the waiting line has a distribution which is a geometric series, and waiting time has a distribution which is negative exponential, both apart from discontinuities at zero. The last has previously been given by Pollaczek [C. R. Acad. Sci. Paris 236, 578–580 (1953); these Rev. 14, 772] as a special case of his more general theory in which service times as well as inter-arrival times have a general distribution.

Detailed calculations of equilibrium congestion probabilities are given for $s=2$ and 3, and a table gives a comparison for $s=1, 2, 3$ of random and regular arrivals with respect to the probability of not having to wait and to average waiting time (in units of the average service time).

J. Riordan (New York, N. Y.).

Foster, F. G. On the stochastic matrices associated with certain queuing processes. Ann. Math. Statistics 24, 355–360 (1953).

A number of general theorems on the behavior of an irreducible Markov chain, either taken from or in the mood of Feller [An introduction to probability theory and its applications, Wiley, New York, 1950; these Rev. 12, 424], are stated and, when necessary, proved. They are then applied to the chains used by Kendall (in the paper reviewed above) verifying and refining the results he has reached.

J. Riordan (New York, N. Y.).

Kallianpur, G., and Robbins, H. Ergodic property of the Brownian motion process. Proc. Nat. Acad. Sci. U. S. A. 39, 525–533 (1953).

Let $\{V(t), 0\leqq t<\infty\}$ be the vector random variables of the plane Brownian motion, and suppose that f and g are bounded and summable on the plane, with integrals $\bar{f}$ and $\bar{g}$ over the plane. Then it is proved that, if $\bar{f}\neq 0$,

$$\lim_{T\to\infty} \Pr\left[\frac{2\pi}{\bar{f}\log T}\int_0^T f[V(t)]dt\leqq u\right]=\max\,[1-e^{-u}, 0],$$

and that, if $g \neq 0$,

$$\lim_{T\to\infty} \frac{\int_0^T f[V(t)]dt}{\int_0^T g[V(t)]dt} = \frac{\bar{f}}{\bar{g}}$$

in probability. [It has since been observed that the second result can be derived from the ergodic theorem for spaces with infinite-valued measures, so that f, g need not be bounded, and the limit is a probability 1 limit.] The one-dimensional version of these theorems is also given.

J. L. Doob (Urbana, Ill.).

Sarymsakov, T. A. On the ergodic principle for nonstationary Markov chains. Doklady Akad. Nauk SSSR (N.S.) 90, 25–28 (1953). (Russian)

Let $\|p_{ij}(1)\|$, $\|p_{ij}(2)\|$, $\cdots$ be the successive one-step transition probability matrices of a Markov chain with s states, and let $p(m, n, i, j)$ be the (i, j)th element of $\prod_{k=m}^{n}\|p_{ij}(k)\|$. A stochastic matrix is called primitive if some power has only positive elements. Let G be the class of s-dimensional primitive stochastic matrices A with the property that AB is primitive whenever B is. Then, if it is possible to find a sequence of matrices in G, $\{\|q_{ij}(k)\|, k \geqq 1\}$, such that, for some $c>0$, and all i, j, k with $q_{ij}(k)>0$, the inequality $p_{ij}(k)>c$ is satisfied, it follows that

$$\lim_{n\to\infty} [p(m, n, i, j) - p(m, n, i', j)] = 0$$

for all m, i, i', j. Related results are also proved.

J. L. Doob (Urbana, Ill.).

Davis, R. C. On the Fourier expansion of stationary random processes. Proc. Amer. Math. Soc. 4, 564–569 (1953).

L'auteur considère une fonction aléatoire réelle $X(t)$ stationnaire du second ordre et continue en m.q., avec $E[X(t)]=0$, $E[X(t)X(t+h)]=r(h)$ ($r(h)$ est continue), et sa série de Fourier $\sum_n A_n e^{2\pi int/T}$ relative à l'intervalle $(0, T)$, avec $A_n = T^{-1}\int_0^T X(t)e^{-2\pi int/T}dt$ en m.q.; il montre que pour que l'on ait, quelque soit T, $E[A_n\bar{A}_m]=0$ quels que soient n et m ($n \neq m$), il faut et il suffit que $X(t)$ se réduise à une constante $[r(h)=r(0)]$.

R. Fortet (Paris).

Kawata, Tatsuo. Stationary process and harmonic analysis. Kōdai Math. Sem. Rep. 1953, 41–60 (1953).

Expository paper. *J. L. Doob* (Urbana, Ill.).

van Woerkom, A. J. J. On cumulative sums of random numbers. Astr. J. 58, 10–20 (1953).

The title of the paper does not exactly correspond to its subject which is to investigate whether or not, as suggested by D. Brouwer [Astr. J. 57, 125–146 (1952)], the fluctuations of the moon's mean longitude can be represented as the sum of a secular trend (a quadratic in time, t) and of an integrated Wiener process. The rationale of this latter component is that, owing to various circumstances, the rotation of the earth may have a component with random mutually independent increments. After applying various methods of analysis, including the correlograms, the author concludes that, although there is a general agreement between the theory and the empirical data, the original theoretical set-up appears too simple to account for all the details of the observations. In particular, it appears necessary to take into account the fact that, in addition to the elements of randomness mentioned above, the observations are affected by a random error independent of the value of the measured magnitude. Furthermore, there are indications that either those random errors or the random increments in the velocity of the earth's rotation, or both, may be auto-correlated. The interesting point in the paper, also present in the above paper by Brouwer, is that, in parallel with the analysis of the actual observations on the moon's mean longitude, there is an analysis of "synthetic" data produced by a large scale sampling experiment giving several sample curves of an integrated Wiener process. Although the theory of stochastic processes has substantial literature, the properties of sample curves produced by the various possible machineries are still far from being intuitively familiar and experiments of the kind described are likely to appeal to the intuition and to contribute to the development of probabilistic theories of physical phenomena. In particular, the author's experiments illustrate the fact that the sample curves of the integrated Wiener process may strongly suggest the presence of a non-random trend when, in fact, none exists.

J. Neyman (Berkeley, Calif.).

Friede, Georg. Invariante Leistungssysteme. Bl. Deutsch. Ges. Versicherungsmath. 1, no. 2, 51–67 (1 plate) (1951).

Consider a group of individuals all of whom effect an insurance simultaneously at age x, and pay a premium P_t at time t ($t=0, 1, 2, \cdots$). These individuals are subject to k decremental forces, the probability of being removed by the jth of them between time t and $t+1$ being $q_t^{(j)}$ ($j=1, 2, \cdots, k$). On removal, the insured is paid a sum $S_tR_t^{(j)}$. The author determines the "invariant" series S_t ($t=0, 1, 2, \cdots$) which leaves the mean value of a policy at time t ($t=0, 1, 2, \cdots$) unaltered when the set $\{q_t^{(j)}, R_t^{(j)}\}$ is changed to the set $\{q_t'^{(j)}, R_t'^{(j)}\}$.

H. L. Seal.

Mathematical Statistics

Matsumura, Soji. Bemerkung zu Korrelations Theorie. J. Osaka Inst. Sci. Tech. Part I. 3, 33–34 (1951).

Stuart, A. The estimation and comparison of strengths of association in contingency tables. Biometrika 40, 105–110 (1953).

The author follows up a suggestion of M. G. Kendall that his rank correlation coefficient may be calculated for a contingency table in which the rows and columns are ordered by the criteria of classification. He introduces a modified form which in the universe of all permutations of the observations can sometimes attain the values ± 1; if the number of observations is large, it can generally almost attain these values. He derives an upper bound for the variance of the measure of association so obtained. He illustrates its calculation and the use of the resulting conservative significance test.

C. C. Craig (Ann Arbor, Mich.

Mardessich, Bartolo. Sulle relazioni fra medie combinatorie e medie potenziate. Statistica, Bologna 13, 77–85 (1953).

Given the n quantities $a_1, a_2, \cdots, a_n$, define

$$S_{k_1,k_2,k_3} = a_1^{k_1}a_2^{k_2}a_3^{k_3} + a_1^{k_1}a_2^{k_2}a_4^{k_3} + \cdots + a_{n-2}^{k_1}a_{n-1}^{k_2}a_n^{k_3},$$

the symmetric function of the third order, and similarly $S_{k_1,k_2,\cdots,k_s}$, the symmetric function of the sth order. If $k_1=k_2=\cdots=k_i=k$, then $S_{k,k,\cdots(i\text{ terms})}$ is denoted by ${}_kS_i$.

The power average is defined by $\{S_k/n\}^{1/k}$, and the combinatorial average by $\{{}_ks_i \div \binom{n}{i}\}^{1/ki}$. The author shows that any symmetric function of the sth order is expressible in terms of symmetric functions of the first order and gives various relations which express ${}_ks_i$ in terms of the S functions.
L. A. Aroian (Culver City, Calif.).

Azorín, F. On the noncentral t distribution. Revista Acad. Ci. Madrid **46** (1952), 491–495 (1953). (Spanish)

An expression is obtained for the frequency functions for non-central t in terms of the moments of a truncated normal distribution. Using the joint distribution of the numerator and denominator the author finds the moments of non-central t. Then using the method discussed by Curtiss [Ann. Math. Statistics **14**, 107–122 (1943); these Rev. **5**, 128] he undertakes to find a transformation of non-central t which is approximately normally distributed. However, some misprints mar his calculations here. *C. C. Craig.*

Gronow, D. G. C. Non-normality in two-sample t-tests. Biometrika **40**, 222–225 (1953).

The author considers two statistics for judging of the significance of a difference between the means of two samples of unequal size drawn from populations with unequal means and variances. He computes the first four moments of a suitable function of the given statistics, and fits them with a Johnson type curve, comparing the result with the t distribution. If the sample sizes are nearly equal, the power of the test is not affected very much either by difference in the variances of the parent populations or by their kurtosis. If the sample sizes differ substantially, difference in the population variances has a considerable effect on the power of the tests, but kurtosis in the populations does not. *A. Blake* (Buffalo, N. Y.).

Olekiewicz, M. On certain improved estimates of the mean. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **5** (1951), 139–146 (1953). (Polish and Russian summaries)

Let a be an estimate of the parameter α. The author calls $a=a_0$ best if $E(a_0-\alpha)^2=\text{minimum}$. "Best" estimates are obtained in various special cases by determining a constant A such that $a_0=Aa$, a being the generally used estimate of α. It turns out that in many cases A depends on nuisance parameters (including α). In these cases, the author suggests various methods for getting around this difficulty, e.g., by determining upper or lower bounds for the nuisance parameters in some way or other. *G. E. Noether.*

Wilson, Edwin B. Significance levels for a skew distribution. Proc. Nat. Acad. Sci. U. S. A. **39**, 537–546 (1953).

In this paper the author discusses several conventions for establishing levels of significance of a random variable x having an arbitrary unimodal probability density function $f(x)$. In particular, by regarding some specific value of x as a "target value" T, he considers sets (intervals) of significant values of x for a given level of significance P of the following forms:

(a) the set of values of x for which $\Pr(|x-T|>h_P)=P$;
(b) the set of values of x for which

$$\Pr(x<x_P)=\Pr(x>x'_P)=\tfrac{1}{2}P;$$

(c) the set of values of x in which

$$\Pr(x<x_P)=\Pr(x>x'_P)=\text{P},$$

where $f(x_P)=f(x'_P)$;

(d) the set of values of x for which

$$\Pr(x<x_P)+\Pr(x>x'_P)=P,$$

where $\Pr(x<x_P)/\Pr(x<T)=\Pr(x>x'_P)/\Pr(x>T)$.

The author shows some preference for (d). In cases (b) and (c), the "target value" T is considered to be the median and the mode of the distribution of x, respectively. The author does not attempt to discuss these conventions in terms of the modern theory of testing statistical hypotheses. Several illustrative examples are presented.
S. S. Wilks (Princeton, N. J.).

Owen, Donald B. A double sample test procedure. Ann. Math. Statistics **24**, 449–457 (1953).

Three double sampling procedures for testing the mean of a normal population with known standard deviation are given for which the power functions are derived and tabulated. A double sampling procedure for Student's hypothesis is also discussed. *R. P. Peterson.*

Aoyama, Hirojiro. On a test in paired comparisons. Ann. Inst. Statist. Math., Tokyo **4**, 83–87 (1953).

Consider two people of two different occupations. If each one rates his occupation higher than the occupation of the other a value of $+1$ is assigned with probability 1/4; if each rates his occupation lower than the occupation of the other a value of -1 is assigned with probability 1/4; and otherwise a value of 0 is assigned with probability 1/2. Consider n people with n different occupations, and make all $\binom{n}{2}$ comparisons. Let the statistic S equal the sum of the values of the $\binom{n}{2}$ comparisons. The author determines the mean, variance, the probability function of S and proves its asymptotic normality for large n. A short table is given for the probability that $|S|\geqq S_0$ for $S_0=0(1)36$, $n=3(1)9$ to five decimals or more. The operating characteristic function is found for the alternative when $+1$ is assigned a probability of p, -1 a probability q and 0 a probability of r.
L. A. Aroian (Culver City, Calif.).

van der Waerden, B. L. Ein neuer Test für das Problem der zwei Stichproben. Math. Ann. **126**, 93–107 (1953).

Let r stand for the ranks of the g observations from the first population in the overall ranking of the $g+h=n$ observations from both (continuous) populations. Let further $\psi(u)$ denote the inverse of the standard normal cumulative distribution function. The author suggests the use of the test statistic $X=\sum_1^g\psi(r/(n+1))$. It is shown that under the null hypothesis of equal populations X is asymptotically normally distributed with mean 0 and variance $gh/[n(n-1)]\sum_1^n\psi(i/(n+1))^2$ provided g is held fixed as $n\to\infty$. [Reviewer's remark: The asymptotic normality of X when g is not held fixed, conjectured by the author, follows from a theorem by Wald and Wolfowitz [Ann. Math. Statistics **15**, 358–372 (1944); these Rev. **6**, 163].] It is further shown that under the same conditions the X-test is asymptotically as powerful as the t-test if under the alternative hypothesis the two populations are normal with equal variance. It is pointed out that in the non-normal case the X-test may be more powerful than the t-test. *G. E. Noether.*

Lehmann, E. L., and Stein, C. M. The admissibility of certain invariant statistical tests involving a translation parameter. Ann. Math. Statistics **24**, 473–479 (1953).

The authors give conditions for the admissibility of the most powerful invariant test for testing one location parameter family against another. *E. Lukacs.*

Birnbaum, Z. W. On the power of a one-sided test of fit for continuous probability functions. Ann. Math. Statistics **24**, 484–489 (1953).

If $F(x)$ is a continuous distribution function (d. f.) of a random variable X, and if $F_n(x)$ is the empirical d. f. determined by a random sample $x_1, x_2, \cdots, x_n$, then $\Pr(F(x) \leqq F_n(x)+\epsilon$, for all $x)$ is known to be a function $P_n(\epsilon)$, independent of $F(x)$. A closed expression for $P_n(\epsilon)$, and some of its values are given in a recent paper by the author and Tingey [same Ann. **22**, 592–596 (1951); these Rev. **13**, 367]. $P_n(\epsilon)$ is used to test the hypothesis $F(x)=G(x)$. Sharp upper and lower bounds are obtained for the power of the test if $\sup_{-\infty<x<\infty} \{H(x)-G(x)\}=\delta$, for preassigned δ. Such bounds are useful, since the exact formula for the power given in the paper will in general require extensive numerical integration. *B. Epstein.*

Blackwell, David. Equivalent comparisons of experiments. Ann. Math. Statistics **24**, 265–272 (1953).

Two ways [Bohnenblust, Shapley, and Sherman, unpublished; and Blackwell, Proc. 2d Berkeley Symposium on Math. Statistics and Probability, 1950, Univ. of California Press, 1951, pp. 93–102; these Rev. **13**, 667] of comparing experiments α, β have been proposed. These are denoted by $\alpha \supset \beta$, α is more informative than β and $\alpha \vdash \beta$, α is sufficient for β. It is immediate that $\alpha \vdash \beta$ implies $\alpha \supset \beta$ for a finite number, N, of states of nature. The current work establishes that $\alpha \supset \beta$ implies $\alpha \vdash \beta$ for a finite number of states of nature. This result had been established for experiments with a finite number of outcomes by Stein (unpublished) and Sherman [Proc. Nat. Acad. Sci. U. S. A. **37**, 826–831 (1951); **38**, 382 (1952); these Rev. **13**, 633]. The current proof even in this case is much neater than those cited. It uses the minimax theorem of Bohnenblust, Karlin, and Shapley [Contributions to the theory of games, Princeton, 1950, pp. 181–192; these Rev. **12**, 514] instead of the more laborious procedure of applying duality theory. For the case of $N=2$ the theorem had been established by Blackwell [op. cit.]. The notion of $\alpha \vdash_k \beta$, α more informative than β for k-decision problems, is introduced and given three equivalent formulations. While it is proved that for $N=2$ $\alpha \vdash_2 \beta$ implies $\alpha \vdash \beta$, it is asserted [Stein, unpublished paper] that it is not true in general that $\alpha \vdash_k \beta$ implies $\alpha \vdash_{k+1} \beta$. If for a sample of size 1 at every level t the probability of an error of the second kind with α does not exceed the corresponding probability for β, then the same is true for every sample size. *S. Sherman* (Sherman Oaks, Calif.).

Steinhaus, H. The principles of statistical quality control. Zastosowania Mat. **1**, 4–27 (1953). (Polish. Russian and English summaries)

Expository and polemical article reviewing various ways of formulating the mathematical problem of an "optimum" inspection plan. The discussion revolves about the familiar fact that the expected loss from the application of any given inspection plan depends essentially upon the frequency distribution of quality in the sequence of lots that will be delivered for inspection. *J. Neyman.*

Wünsche, Günther. Sequential-Testverfahren in der Versicherungstechnik. Bl. Deutsch. Ges. Versicherungsmath. **1**, no. 4, 19–37 (1953).

Expository paper. *H. L. Seal* (New York, N. Y.).

Zwinggi, Ernst. Anwendung neuerer statistischer Verfahren in der Versicherungsmathematik. Bl. Deutsch. Ges. Versicherungsmath. **1**, no. 1, 13–23 (1950).

Münzner, H. Statistische Testmethoden in der Versicherung. Bl. Deutsch. Ges. Versicherungsmath. **1**, no. 1, 29–37 (1950).

Miller, K. S., and Schwarz, R. J. On the interference of pulse trains. J. Appl. Phys. **24**, 1032–1036 (1953).

Given the period and the duration of the pulses of two pulse trains the authors consider the following problems. (a) What average fraction of time are the pulses of two pulse trains coincident for two given initial phases. (b) What is the expected value of this fraction of time when the initial phases have a rectangular distribution. (c) What average fraction of the time are the pulses coincident during a time interval at least equal to a given length when the initial phases are fixed. (d) What is this expected value if the initial phases have a rectangular distribution. The authors solve these four problems for the case that the periods, durations and initial phases are commensurable. In this case the problems lead to certain sets of linear congruences and the solution to (a) and (c) are expressed in terms of the number of congruences in this set which have a solution. The solutions to (b) and (d) are given explicitly. *H. B. Mann.*

Mathematical Economics

von Mises, Richard. Über die J. von Neumannsche Theorie der Spiele. Math. Nachr. **9**, 363–378 (1953).

The author proves that every real matrix contains square submatrices of what he calls "normal type", i.e., the sums of the rows and the sums of the columns of the algebraic complements of the various elements do not all vanish, and those which do not vanish have the same sign. Among these square submatrices of normal type there is at least one which satisfies certain other conditions not easily stated here. The author shows that from such a one it is easy to obtain the value and an optimal strategy for each player of a zero-sum two-person game, of which the original matrix is the pay-off matrix. Every extreme point of the convex set of optimal strategies can be obtained in this manner. *J. Wolfowitz* (Ithaca, N. Y.).

***Guilbaud, Georges Th. The theory of games. Critical contributions to the theory of value.** International Economic Papers, No. 1, pp. 37–65. The Macmillan Co., London-New York, 1951.

Translated from Economie Appliquée **2**, 275–319 (1949).

***Zaccagnini, Emilio. Simultaneous maxima in pure economics.** International Economic Papers, No. 1, pp. 208–244. The Macmillan Co., London-New York, 1951.

Translated from Giornale degli Economisti e Annali di Economia (N.S.) **6**, 258–292 (1947).

***Dantzig, George B. Maximization of a linear function of variables subject to linear inequalities.** Activity Analysis of Production and Allocation, pp. 339–347. Cowles Commission Monograph No. 13. John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1951. $4.50.

This paper contains the first published version of the "simplex method", developed by the author in 1947. It is still (in 1953) the best computational procedure for solving the following problem, basic to linear programming and matrix games: Find the values of λ_j which

maximize the linear form $c_1\lambda_1+\cdots+c_n\lambda_n$ subject to $\lambda_j \geqq 0$ and $a_{i1}\lambda_1+\cdots+a_{in}\lambda_n=b_i$ where a_{ij}, b_i, c_j are constants ($i=1, \cdots, m$ and $j=1, \cdots, n$ throughout). The method can be described most concisely by a geometric model in $(m+1)$-dimensional space with coordinates $x_1, \cdots, x_m, z$. If P_j denotes the point with coordinates $(a_{1j}, \cdots, a_{mj}, c_j)$ then the problem is: Find the points with greatest z-coordinate in the convex polyhedral cone spanned by the P_j and on the line $L: x_1=b_1, \cdots, x_m=b_m$. A feasible basis is a set of m of the points, say $P_1, \cdots, P_m$, that span an m-dimensional convex cone intersected by L at $z=z_0$. If another point, P_j, lies above the hyperplane spanned by $P_1, \cdots, P_m$ then one of these points is replaced by P_j to form a new feasible basis with a higher value of z_0. Otherwise the given basis provides a solution. This description neglects questions of existence and degeneracy which are treated by the author.

H. W. Kuhn (Bryn Mawr, Pa.).

***Dorfman, Robert. Application of the simplex method to a game theory problem.** Activity Analysis of Production and Allocation, pp. 348–358. Cowles Commission Monograph No. 13. John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1951. $4.50.

This paper contains an application of the "simplex method" of G. B. Dantzig (see the review above) to the solution of an arbitrary finite zero-sum two-person game by transforming the game into an equivalent linear program. In this special case, the presence of an immediate feasible basis facilitates computation. A 5×6 matrix game is solved as an illustration and it is remarked that the method yields optimal mixed strategies simultaneously for both players.

H. W. Kuhn (Bryn Mawr, Pa.).

***Dantzig, George B. Application of the simplex method to a transportation problem.** Activity Analysis of Production and Allocation, pp. 359–373. Cowles Commission Monograph No. 13. John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1951. $4.50.

In this paper, the author applies his "simplex method" (see the review second above) to the Hitchcock-Koopmans "transportation problem": A homogeneous product is to be shipped in amounts a_i from m shipping origins i and received in amounts b_j by each of n shipping destinations j. The cost of shipping a unit amount of the product from origin i to destination j is c_{ij}. Find the amounts x_{ij} to be shipped from i to j so as to minimize the total cost. An example with $m=3$ and $n=5$ is solved as an illustration.

H. W. Kuhn (Bryn Mawr, Pa.).

***Charnes, A., Cooper, W. W., and Henderson, A. An introduction to linear programming.** John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London, 1953. x+74 pp. $2.50.

This volume is composed of two parts: an economic introduction to linear programming and a set of lectures on its mathematical background. In the first part, Cooper and Henderson present the complete solution of a numerical example by the "simplex method" of G. B. Dantzig (see the review third above). This section is remarkable for the careful explanations of the computational detail accompanied by trenchant economic explanations at many points. In sum, most of the features of linear programming are illuminated by a deceptively simple example. In the second part, Charnes develops chapters from the various mathematical disciplines germane to linear programming, concentrating again on the simplex method; the high quality of exposition is maintained in this section. Methods originated by the author for handling degeneracy and for developing the full set of solutions are included, as is a short treatment of the subject of duality.

The parts may be read independently, but have been cross-referenced to serve the needs of readers with varying backgrounds.

H. W. Kuhn (Bryn Mawr, Pa.).

Charnes, A. Optimality and degeneracy in linear programming. Econometrica **20**, 160–170 (1952).

A computational procedure is given for solving the problem of linear programming in the following form: Find $\lambda=(\lambda_1, \cdots, \lambda_n)$, $\lambda_i \geqq 0$, such that $\sum_{i=1}^n \lambda_i P_i = P_0$ and $f(\lambda)=\sum_{i=1}^n c_i\lambda_i$ is a maximum, where the P_i's are vectors in m-space. The method is an improvement on the simplex method of Dantzig (see the paper reviewed fourth above). The improvement consists in eliminating the possibility of degeneracy (linear dependence) which may cause the original simplex method to break down. The algorithm also provides a means of obtaining all solutions of the programming problem.

D. Gale (Copenhagen).

***Brown, George W. Iterative solution of games by fictitious play.** Activity Analysis of Production and Allocation, pp. 374–376. Cowles Commission Monograph No. 13. John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1951. $4.50.

This short note proposes a simple iterative method for approximating the solutions of zero-sum two-person games. At every stage in the iteration, each player chooses a pure strategy that is optimal against the cumulative choices of his opponent to date. The method is notable for the simplicity of its computational routine. It has since been shown to converge by J. Robinson [Ann. of Math. (2) **54**, 296–301 (1951); these Rev. **13**, 261].

H. W. Kuhn.

Marshall, A. W. A mathematical note on sub-optimization. J. Operations Res. Soc. Amer. **1**, 100–102 (1953).

This expository note gives a clear demonstration of the well-known theorem on the optimal allocation of resources in production: an allocation of resources to the production of different commodities is efficient (in the sense that no reallocation of resources will increase the outputs of all commodities) if the marginal rate of substitution between each pair of resources is the same in the production of all commodities.

K. J. Arrow (Stanford, Calif.).

Malinvaud, Edmond. Capital accumulation and efficient allocation of resources. Econometrica **21**, 233–268 (1953).

The author has given an admirable restatement of capital theory as a problem in the optimal allocation of resources over time. For each time period t, let b_t be the vector of commodities produced at time t, a_t the vector of inputs, x_t the vector of goods consumed and labor services offered by households, $y_t=a_t-b_t$, and $\bar{z}_t$ is the flow of natural resources. Clearly, $x_t+y_t \leqq \bar{z}_t$ for all t. The initial value b_1 is regarded as given. Finally, production is regarded as taking one time period; the output b_{t+1} is regarded as limited by the input a_t. More precisely, the technological possibilities of production at time t are represented by a set T_t, such that the pair $(a_t, b_{t+1}) \in T_t$ for all t. In this model, durable producers' goods enter as both input and output. Under the usual assumptions of divisibility and additivity of production processes, the set T_t is a closed convex cone; for

most of the results of the paper, the set need be only closed and convex. A sequence x^1 of vectors x_t^1 is efficient if there is no other sequence satisfying the above conditions such that $x_t \geqq x_t^1$ with the strict inequality holding for at least one commodity in one time period. The chief aim of the paper is to characterize efficient sequences (here called "chronics") in terms of prices. If the time period considered is finite, the problem is the classical one of welfare economics; if the same commodity at two different points of time is regarded as two distinct commodities, we have only a finite number of commodities altogether, and an efficient chronic is characterized by a set of associated prices such that the chronic in question maximizes profits compared with any other chronic.

If the index t runs over an infinite set, there are convergence difficulties. The following generalization of the pricing theorem for efficient chronics is stated by the author: If x is an efficient chronic, there is a non-negative sequence p of vectors p_t such that, for all h, $\sum_{t=1}^{h} p_t y_t$ is minimal among all chronics such that $x_t^1 = x_t$ for $t > h$. (From the context it is clear that the term "nonnegative" is meant to imply that at least one component is positive.) The proof given appears to be defective, and it is not easy to see how to repair it along the lines now given. However, a valid proof can be given by a modification of the author's reasoning for the case where the sets T_t are bounded by differentiable surfaces.

A partial converse to the above theorem is also given, and the applications of the prices obtained to decentralization of production and to the formulation of the rules for profit-maximization for individuals and firms are studied. The theory of interest arising from this model and the relations with monetary rules are discussed, as well as applications to national income accounting. A great number of troubling points in the usual verbal discussions are easily resolved in the present general model. Finally, the special case of stationary chronics, which has been much discussed in the literature, is taken up in some detail. *K. J. Arrow.*

Allais, M. L'extension des théories de l'équilibre économique général et du rendement social au cas du risque. Econometrica 21, 269–290 (1953).

The author studies the theories of general economic equilibrium and of optimal allocation of resources when some of the commodities traded in are random variables. He considers first the case where all risks are real ones for society as a whole (as opposed to artificial gambling devices) and there are no insurance companies or other means of transforming risks from one type to another. To simplify the problem, he supposes two commodities, one (A) a sure good, and one (B) whose unit is a random variable normally distributed with an expectation which is one unit of A and a standard deviation σ. Each separate unit is assumed to give rise to an independent random variable, so that the variance of B units of the second commodity is $\Sigma = \sigma\sqrt{B}$. The satisfaction derived from having A units of the first commodity and B of the second is a function of A, B, and Σ; the author assumes this function to be of the form $S(A, B+\lambda\Sigma)$, where λ is positive if risk is preferred and negative if security is preferred. (The expected-utility formulation of choice under uncertainty is explicitly rejected.) It is then shown that the satisfaction can be taken equal to $A+B+\lambda\sigma\sqrt{B}$. Let A^0 and B^0 be the initial holdings of the two commodities by the individual and let a and b be their prices, respectively. The individual then acts so as to maximize satisfaction subject to the restraint $aA+bB = aA^0+bB^0$. He notes that the maximum occurs at a corner (either $A=0$ or $B=0$) unless $b>a$ and $\lambda>0$ both hold. In that case, there is a tangential maximum defined by $\sqrt{B} = \lambda\sigma/[2(b/a-1)]$. (The author has failed to observe that even in this case there may be a corner maximum; it may well happen that the above value of B, when substituted into the budget restraint, implies a negative value of A.)

With the aid of these results, the market equilibrium for a number of individuals is discussed. If all individuals prefer risk ($\lambda_p > 0$ for all individuals p), then $b/a > 1$; if each individual has the same value of λ_p and all maxima are tangential, then $b/a = 1+\frac{1}{2}\lambda\sigma(B/n)^{-1/2}$; b/a approaches 1 if the average quantity of the chance good approaches zero. If $\lambda_p < 0$ for all p, then at equilibrium $b/a < 1$; the equilibrium may be somewhat indeterminate, but the range of indeterminacy will be small.

The author next considers the introduction of lotteries and shows that if all individuals prefer risk, the ratio b/a will decline and all individuals will be better off. Similarly, if all individuals prefer security, the introduction of insurance companies and other methods of consolidating risks will raise b/a closer to 1 and make everyone better off.

The extension of the usual theorems of welfare economics to the case of risk is next considered. It is argued that as usual the competitive equilibrium is a necessary and sufficient condition for an optimal allocation of resources in the sense of Pareto. (The necessity argument appears incorrect for the case where all individuals prefer security, i.e., there may exist optimal allocations which cannot be realized by a competitive system. Suppose there are two individuals, with $\lambda_1 = -1$ and $\lambda_2 = -2$, initial supplies of both commodities are 2, $\sigma = 1$, and we seek to find that allocation which maximizes the satisfaction of individual 1 subject to the condition that the satisfaction of individual 2 be at least 1. It is easy to see that $A_1 = 1$, $B_1 = 2$; but this is not a corner maximum and it has been shown previously that any competitive equilibrium for the case where there is preference for security must be attained at a corner maximum. The trouble is that the satisfaction function is not quasi-concave in A and B.)

Relations with social policy are briefly considered.

K. J. Arrow (Stanford, Calif.).

Solow, Robert M., and Samuelson, Paul A. Balanced growth under constant returns to scale. Econometrica 21, 412–424 (1953).

The authors consider the problem of balanced growth in a model where there is joint production with continuously variable proportions, but where the inputs uniquely determine the outputs taken as a whole. If $a_1, \cdots, a_n$ are the inputs in one time period, the outputs in the next time period are $H^i(a_1, \cdots, a_n)$ $(i=1, \cdots, n)$, all assumed homogeneous of the first degree, continuous, and strictly monotonic in each variable. Labor is included among the outputs, being produced by the activity of consumption, so that the entire output vector is simultaneously an input vector. If $X_i(t)$ is the output of commodity i at time t, then $X_i(t+1) = H^i[X_1(t), \cdots, X_n(t)]$ for all i and t. By balanced growth is meant a solution to this system of difference equations in which the ratios among the different outputs is constant over time, i.e., $X_i(t+1) = \lambda X_i(t)$ for all i and t and some positive λ. Hence, $X_i(t) = xV_i\lambda^t$, where x and $V_1, \cdots, V_n$ are positive, and $\lambda V_i = H^i(V_1, \cdots, V_n)$ $(i=1, \cdots, n)$, where, by suitable choice of x, it can be

assumed that $\sum_{i=1}^{n} V_i = 1$. The existence of $\lambda, V_1, \cdots, V_n$ satisfying the above conditions is then demonstrated with the aid of Brouwer's fixed point theorem; further, the values are unique. From a theorem of Krein and Rutman, a lower bound on λ can be given; i.e., if there exist $V_1, \cdots, V_n$ such that for each i, $H^i(V_1, \cdots, V_n) \geqq cV_i$, then $\lambda \geqq c$.

The next question is the stability of the proportions, i.e., will an arbitrary solution of the difference equations converge to a balanced growth path with rate of expansion and proportions $V_1, \cdots, V_n$. The answer is affirmative in all cases. Since stability in the large implies stability in the small and the latter condition can be expressed in terms of bounds on the characteristic roots of certain matrices, some matrix theorems are obtained as by-products.

Some analogous problems for differential equations are also considered. *K. J. Arrow* (Stanford, Calif.).

Eyraud, Henri. Transfiscalité et rétrofiscalité. Ann. Univ. Lyon. Sect. A. (3) **15**, 37–45 (1952).

Consider a monopolist with a linear cost curve, selling to a dealer whose handling costs are also linear and who in turn faces a linear market demand. In this simple setup the author shows that the incidence, on price, output, and the dealer's and producer's profits, of a specific tax is the same whether the tax is levied on producer, dealer, or consumer. He considers also the rate of tax which maximizes the yield to the state, and points out that if taxes are levied simultaneously at all levels, maximizing their separate yields is not the same as maximizing their joint aggregate yield. If a proportionate tax on profits is combined with a specific tax, once the rate of the former exceeds a certain level, maximum joint yield would require the latter to be turned into a subsidy. In equation (7) f is a misprint for r. *R. Solow.*

Eyraud, Henri. Théorie mathématique des changes. Ann. Univ. Lyon. Sect. A. (3) **15**, 47–54 (1952).

A typical problem studied is the following: there are two countries, each producing a different product with a linear cost curve and each having a linear demand for the other's product. Assume that each country independently maximizes its net revenue from trade, treating the other's demand curve as given. The two demand functions, the two equations of profit maximization, and an equation stating that international payments (including given unilateral transfers or debt repayments) must balance, together determine outputs, domestic-currency prices, and the exchange rate. This extends to more countries in the obvious way.

R. Solow (Cambridge, Mass.).

Scott, A. D. Bibliography of applications of mathematical statistics to economics. Supplement for 1950. J. Roy. Statist. Soc. Ser. A. **116**, 177–185 (1953).

For a bibliography for the years 1943–1949 see the same J. Ser. A. **114**, 372–393 (1951); these Rev. **13**, 370.

TOPOLOGY

Bäbler, F. Über eine spezielle Klasse Euler'scher Graphen. Comment. Math. Helv. **27**, 81–100 (1953).

The author studies the "arbitrarily traceable" graphs introduced by Ore [Elemente der Math. **6**, 49–53 (1951); these Rev. **12**, 845]. He investigates the number of ways of decomposing an arbitrarily traceable graph into circuits no two of which have a common edge. He gives a necessary and sufficient condition for the existence of an arbitrarily traceable graph having a given number of such decompositions. The author uses the theory of arbitrarily traceable graphs to give another proof of a theorem of Senior [Amer. J. Math. **73**, 663–689 (1951); these Rev. **13**, 147]. This theorem gives a necessary and sufficient condition for the existence of a connected loopless graph with a specified number of vertices of each degree. *W. T. Tutte.*

Ungar, Peter. On diagrams representing maps. J. London Math. Soc. **28**, 336–342 (1953).

The author shows that for each map in the plane having only a finite number of regions, all simply connected, and such that just three regions meet at each vertex there exists an isomorphic planar map in which each region is a rectangle or the exterior of a rectangle.

W. T. Tutte (Toronto, Ont.).

Sedmak, Viktor. Quelques applications des ensembles partiellement ordonnés. C. R. Acad. Sci. Paris **236**, 2139–2140 (1953).

The author announces extensions of his previous results on the ordinal dimension of polyhedra [Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **7**, 169–182 (1952); these Rev. **14**, 783]. Theorem 1. The dimension of a 3-dimensional polyhedron P, denoted by $d_0[P]$, is $\geqq 4$ for all such polyhedra. Theorem 2. The supremum of $d_0[P]$ is $\aleph_0$, over all 3-dimensional polyhedra. A detailed exposition is promised, to appear in another periodical.

E. Hewitt (Seattle, Wash.).

Smirnov, Yu. M. On metrization of topological spaces. Amer. Math. Soc. Translation no. **91**, 17 pp. (1953).

Translated from Uspehi Matem. Nauk (N.S.) **6**, no. 6(46), 100–111 (1951); these Rev. **14**, 70.

de Groot, J. Decomposition spaces. I. Nederl. Akad. Wetensch. Proc. Ser. A. **54** = Indagationes Math. **13**, 109–115 (1951).

The author calls a space M separable if M is a separable metrizable space. Given a space M and a decomposition of M into disjoint closed sets, it is possible to define a decomposition space M^*. The paper contains the theorem that if M is separable then M^* is separable if and only if M^* is regular and satisfies the first axiom of countability. The reviewer was unable to follow the proof because of the use of the word "neighborhood" in two non-equivalent ways.

E. E. Floyd (Charlottesville, Va.).

Saito, Shiroshi. Retracts in the locally compact Hausdorff spaces. Mem. Fac. Sci. Kyūsyū Univ. A. **6**, 157–166 (1952).

A study of absolute (neighborhood) retracts for the case where the spaces involved are locally compact Hausdorff. Most of the basic theorems are shown to extend to this case, but it is asserted that the sum theorem holds only in a weak form, in a sense to be discussed in a later paper, and that local contractibility does not characterize absolute neighborhood retracts. *E. G. Begle* (New Haven, Conn.).

Noguchi, Hiroshi. A generalization of absolute neighborhood retracts. Kōdai Math. Sem. Rep. **1953**, 20–22 (1953).

A separable metric space X is an ϵ-retract of a space $Y \supset X$ if for every $\epsilon' > 0$ there is a mapping $r_{\epsilon'}$ of Y into X such that $\rho(r_{\epsilon'}(x), x) < \epsilon'$ for each x in X. Using this, ϵ-absolute retracts and ϵ-absolute neighborhood retracts are defined in the obvious way. A number of basic theorems for absolute (neighborhood) retracts are shown to have analogues for ϵ-absolute (neighborhood) retracts. In particular, it is shown that each ϵ-absolute retract has the fixed-point property. *E. G. Begle* (New Haven, Conn.).

Knaster, B. Un théorème sur la compactification. Ann. Soc. Polon. Math. **25** (1952), 252–267 (1953).

Let X be an n-dimensional separable metric space. Then X can be embedded topologically in a euclidean space in such a way that $\dim \bar{X} = n$, and hence so that $\dim_p (\bar{X}) \leq \dim X$ for each point p in $\bar{X} - X$. The question studied here is whether X can be embedded in a compact space X^* in such a way that again $\dim_p (X^*) = \dim_p (X)$ for p in X, but $\dim_p (X^*) > \dim X$ for p in $X^* - X$. A necessary condition for the existence of such an X^* is that X be an absolute G_δ. It is shown that this condition is also sufficient if X can be embedded in a euclidean space in such a way that X is the intersection of a sequence of open subsets of $\bar{X}$ each having a boundary of dimension $n-1$ at most. An absolute G_δ which has this last property is said to be of the first genus, otherwise of the second. Examples of absolute G_δ's of each genus are exhibited and a number of problems concerning this classification of absolute G_δ's are proposed.
E. G. Begle (New Haven, Conn.).

Ellis, D. Orbital topologies. Quart. J. Math., Oxford Ser. (2) **4**, 117–119 (1953).

"A natural question in general topology is: Given a set S and a mapping γ of S into itself, how may one construct a topology in S, in general non-trivial, so that in this topology γ will be continuous? In this note I show how to construct a topology from any such pair S and γ and show that, if γ is finite-to-one, γ is continuous in this topology and that, if γ is bi-uniform and onto S, then γ is actually a homeomorphism of S onto itself in this topology. In the last section of the note I find as an application a theorem asserting the existence of T_1 spaces having unusual groups of homeomorphisms." (The author's Introduction.)
A. D. Wallace (New Orleans, La.).

Borsuk, K. On certain mapping of the 2-sphere onto itself. Ann. Soc. Polon. Math. **25** (1952), 268–272 (1953).

It is well-known that a mapping of a separable metric space X into the circumference S_1 is inessential if and only if the mapping can be factored into a mapping of X into the line followed by a mapping of the line onto S_1. In this note an example is given to show that this theorem does not generalize to higher dimensions. A mapping of the 2-sphere S_2 into itself is defined which is inessential but which cannot be factored into a mapping of S_2 into the plane followed by a mapping of the plane onto S_2.
E. G. Begle (New Haven, Conn.).

Borsuk, K., and Jaworowski, J. W. On labil and stabil points. Fund. Math. **39** (1952), 159–175 (1953).

A point p of a metric space S is homotopically labile if for each $\epsilon > 0$ there is a mapping g of $X \times I$ into X such that $g(x, 0) = x$ for all x in X, $\rho(x, g(x, g(x, t))) < \epsilon$ for all (x, t) in $X \times I$, and $g(x, 1) \neq p$ for all x in X. The point p is labile if for each $\epsilon > 0$ there is a mapping f of X into itself such that $\rho(x, f(x)) < \epsilon$ and $f(x) \neq p$ for all x in X. The point p is (homotopically) stable if it is not (homotopically) labile. Every homotopically stable point is stable, and the converse is true in absolute neighborhood retracts but not in general.

These notions are applied to the study of Cartesian products. It is shown that if (x, y) is (homotopically) stable in $X \times Y$, then x is (homotopically) stable in X. The converse is an open question, but partial results are obtained by showing first that if A is a compact subset of an n-dimensional space X and if a is a point of A which is linked in A with a compact set $A_0 \subset A - a$, then a is homotopically stable in X, and then by showing that if A_0 and B_0 are closed subsets of the compact spaces A and B and if $a \in A - A_0$ is linked with A_0 in dimension k and $b \in B - B_0$ is linked with B_0 in dimension l, then (a, b) is linked with $A_0 \times B \cup A \times B_0$ in dimension $k + l$. *E. G. Begle.*

***Borsuk, Karol. Les transformations en sphères et la théorie de la décomposition des espaces euclidiens.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 363–366. Akadémiai Kiadó, Budapest, 1952. (Hungarian and Russian summaries)

Let S_n be the euclidean n-sphere and A a compact space. Denote by $(S_n{}^A)$ the Borsuk group [C. R. Acad. Sci. Paris **202**, 1400–1403 (1936)] whose elements are the homotopy classes of mappings of A into S_n. The object of this note is to show, using a minimum of algebraic arguments, that the number of bounded components of $E_{n+1} - A$ is equal to the rank modulo m of $(S_n{}^A)$, where m is the rank of $(S_n{}^{S_n})$, and hence that the number of bounded components depends only on the topological structure of A and not on the manner in which it is embedded in E_{n+1}. *E. G. Begle.*

Borsuk, Karol. Concerning the homological structure of the functional space $S_m{}^X$. Fund. Math. **39** (1952), 25–37 (1953).

The Corollary to the main theorem states that if a compact space X of dimension at most k has positive kth Betti number then, for every $m \geq k$, the function space $S_m{}^X$ of mappings of X into the m-sphere S_m has positive $(m-k)$th Betti number. The main theorem is a more general theorem of the same sort involving an arbitrary compact space X and true cycles with arbitrary coefficients. Three problems, suggested by this result, are formulated at the end of the paper. *R. H. Fox* (Princeton, N. J.).

Baum, John D. An equicontinuity condition for transformation groups. Proc. Amer. Math. Soc. **4**, 656–662 (1953).

The principal theorem of this paper is the following: Let (X, T) be a transformation group consisting of a compact uniform space X and an abelian topological group T acting on X such that X is minimal under T, that is, the orbit closure under T of every point of X is X itself. Then T is equicontinuous if and only if for every continuous real-valued function f on X there exists a point x of X such that $f(xt)$ is an almost periodic function on T. This theorem extends an unpublished theorem of Kakutani which concerns a continuous flow in a compact metric space.

The author's proof seems to be in need of emendation. The second sentence of page 662 appears to be at fault because Lemma 3.6 should refer to a finite number of functions $f_1, \cdots, f_n$ rather than to just two functions f and g.

The proof of the emended lemma would proceed more perspicuously by considering the orbit closures of the functions $f_1(xt), \cdots, f_n(xt)$ which are compact, whence their union A is compact and the equicontinuous functional transformation group is therefore almost periodic on A. With this alteration, the proof of the above theorem is complete.

W. H. Gottschalk (Philadelphia, Pa.).

Nikaidô, Hukukane. Zum Beweis der Verallgemeinerung des Fixpunktsatzes. Kōdai Math. Sem. Rep. **1953**, 13–16 (1953).

A minor modification of the original proof of a fixed point theorem by Ky Fan [Proc. Nat. Acad. Sci. U. S. A. **38**, 121–126 (1952); these Rev. **13**, 858]. *E. G. Begle.*

Sakai, Shozo. On the map excision theorem. Sci. Rep. Tokyo Bunrika Daigaku. Sect. A. **4**, 290–297 (1953).

Theorem. Let X and Y be spaces, let A and B be closed sets in X and Y whose boundaries lie in open sets with fully normal closures and let $f\colon (X, A)\rightarrow(Y, B)$ be a closed continuous function taking $X-A$ topologically onto $Y-B$. Then the induced homomorphism $f^*\colon H^p(Y, B)\rightarrow H^p(X, A)$ is an isomorphism onto. This improves a result due to Wallace [Duke Math. J. **19**, 177–182 (1952); these Rev. **13**, 765]. It should be noted that Dowker [Ann. of Math (2) **56**, 84–95 (1952); these Rev. **13**, 967] has shown that the two types of groups for which the author validates his results are isomorphic. These are the Alexander-Kolmogoroff-Spanier and the Čech-Dowker groups. *A. D. Wallace.*

***Hilton, P. J. An introduction to homotopy theory.** Cambridge Tracts in Mathematics and Mathematical Physics, no. 43. Cambridge, at the University Press, 1953. viii+142 pp.

In the preface, the author states the purpose of this book as follows. "At the moment, no textbook of homotopy theory exists at any level, with the result that the newcomer to this branch of mathematics is obliged to plunge straight into the study of original papers, often of very considerable complexity. This monograph is designed to fill the gap. It does not claim to be a comprehensive treatment of its subject (the recent work of the French school, for example, is not included, except for a brief introduction to it in the last section of Chapter V); but it is hoped that the reader familiar with Lefschetz's Introduction to Topology [Princeton, 1949; these Rev. **11**, 193] will obtain an understanding of the fundamental ideas of homotopy theory from the first six chapters of this book."

The first chapter is an introduction and contains definitions of some of the basic notions of the subject. Chapter II describes various methods of definition of the absolute and relative homotopy groups, and of the operations of the fundamental groups on the higher homotopy groups. The third chapter, entitled "The classical theorems of homotopy theory", is concerned mainly with the following three: the simplicial approximation theorem, the theorem stating that the homotopy class of a map of an n-sphere onto itself is determined completely by the Brouwer degree, and the Hurewicz theorem that in an $(n-1)$-connected polyhedron the nth homology and homotopy groups are isomorphic. Actually, the chapter contains only the statements and a discussion of these theorems; for the details of the proof, the reader is referred to the text by Lefschetz [op. cit.]. The next chapter contains a description of the homotopy sequence of a pair consisting of a space and a subspace, a proof that this sequence is exact, and some applications of this fact. Chapter V is concerned with homotopy relations in fibre spaces; particular attention is given to the fibre maps of spheres due to Hopf. Chapter VI is about the Hopf Invariant for maps of spheres, the Freudenthal suspension theorems, and various generalizations of these concepts. Most of the more difficult proofs are omitted, the reader being referred to the original papers for full details.

The last two chapters are somewhat different from the first six, being an account of the homotopy theory of complexes. Chapter VII is devoted to a treatment of J. H. C. Whitehead's CW-complexes, while Chapter VIII is concerned with the problem of determining the $(n+1)$-dimensional homotopy group of an $(n-1)$-connected CW-complex $(n>2)$. This problem is considered mainly in order to illustrate the concepts of the previous chapters.

This book will certainly prove very helpful to students trying to learn homotopy theory. Perhaps the most noticeable omission is the lack of an account of the theory of obstructions to extensions and to homotopies of continuous maps. For example, nowhere in the book is Hopf's theorem on the classification of maps of an n-complex into an n-sphere even mentioned. It is also regrettable that the author did not include more discussion to motivate the introduction of the various new ideas and to explain their importance.

W. S. Massey (Providence, R. I.).

Spanier, E. H., and Whitehead, J. H. C. A first approximation to homotopy theory. Proc. Nat. Acad. Sci. U. S. A. **39**, 655–660 (1953).

Let $\mathbf{C}$ be the homotopy category whose objects are pairs (X, x_*), $x_* \in X$, and whose mappings are homotopy classes of maps $X, x_*\rightarrow Y, y_*$. The authors consider a new category, the S-category, $\mathbf{C}_s$, whose objects are as those of $\mathbf{C}$; $\mathbf{C}_s$ may be regarded as an approximation to $\mathbf{C}$ in that there is a natural map of $\mathbf{C}$ into $\mathbf{C}_s$ which under certain circumstances ("in the suspension range" when the spaces are CW-complexes) is an isomorphism. The new category is simpler than $\mathbf{C}$; thus, in $\mathbf{C}_s$, suspension is always an isomorphism.

We now expand the indications given above of the content of the paper. To include relative homotopy (and triad homotopy, etc.), the authors generalize the homotopy problem by considering as objects triples $(X, x_*; \alpha)$, where α is a family of subsets of X such that $(x_*) \in \alpha$ and $A \in \alpha$ implies $x_* \in A$. A carrier $\phi\colon (X, x_*; \alpha)\rightarrow(Y, y_*; \beta)$ is a mapping of α into β such that $\phi x_*=y_*$ and $A\subset A'$ imply $\phi A\subset\phi A'$, $A, A' \in \alpha$. A map $f\colon X\rightarrow Y$ is a ϕ-map if $fA\subset\phi A$, $A \in \alpha$, and a homotopy $f_t\colon X\rightarrow Y$ is a ϕ-homotopy if $f_tA\subset\phi A$, $A \in \alpha$. The collection of ϕ-homotopy classes is written $\pi(\phi)$ and $\mathbf{C}$ is the category whose objects are triples $(X, x_*; \alpha)$ and whose mappings are pairs (ϕ, ρ) where ϕ is a carrier and ρ is a ϕ-homotopy class. Note that $\pi(\phi)$ possesses a distinguished element, the class of the constant map.

Let E' be the interval $-1\leqq t\leqq 1$ and let SX ("suspension of X") be the space formed from $X\times E'$ by identifying $X\times -1\cup X\times 1\cup x_*\times E'$ to a point, which may still be called x_*. Let $S\alpha$ be the family of sets SA, $A \in \alpha$, let $S\phi$ be the carrier: $(SX, x_*; S\alpha)\rightarrow(SY, y_*; S\beta)$, given by $(S\phi)(SA)=S(\phi A)$, and let $Sf\colon SX\rightarrow SY$ be the map induced by the map $(x, t)\rightarrow(fx, t)$ of $X\times E'$ into $Y\times E'$ (recall that $fx_*=y_*$). Then $f\rightarrow Sf$ obviously induces a mapping $\pi(\phi)\rightarrow\pi(S\phi)$. Consider the sequence

$$\Sigma\colon\ \pi(\phi)\rightarrow\pi(S\phi)\rightarrow\cdots\rightarrow\pi(S^n\phi)\rightarrow\cdots.$$

We note first that the set $\pi(S^n\phi)$, $n\geqq 1$, may be given a group structure under track addition which is abelian if $n\geqq 2$.

Moreover, the suspension maps of Σ are homomorphisms. Thus the direct limit of Σ is an abelian group $\pi_s(\phi)$. We call its elements S-ϕ-mappings; thus the $S^m\phi$-map $f: S^mX\rightarrow S^mY$ and the $S^n\phi$-map $g: S^nX\rightarrow S^nY$, m, $n\geqq 0$, determine the same S-ϕ-mapping if and only if there exist integers p, q such that $m+p=n+q$, $=k$, say, and S^pf and S^qg are $S^k\phi$-homotopic. Let $\mathbf{C}_s$ be the S-category whose objects are triples $(X, x_*; \alpha)$ and whose mappings are pairs (ϕ, σ) where ϕ is a carrier and σ an S-ϕ-mapping. Multiplication in $\mathbf{C}_s$ is defined by composition and there is an obvious functor $\mathbf{C}\rightarrow\mathbf{C}_s$. If X is a CW-complex and α a collection of subcomplexes of A, then the mapping $\pi(\phi)\rightarrow\pi_s(\phi)$ is a 1-1 correspondence "within the suspension range", i.e., if $\dim X\leqq 2n-2$, and Y and all $B\in\beta$ are $(n-1)$-connected.

Let ϕ, ψ be carriers $(X, x_*; \alpha)\rightarrow(Y, y_*; \beta)$ such that $\phi A\subset\psi A$, $A\in\alpha$. Then every ϕ-map is a ψ-map and every ϕ-homotopy a ψ-homotopy. There is thus an "injection" $\pi(\phi)\rightarrow\pi(\psi)$. If we define the carrier $\phi_m{}^n$: $(S^mX, x_*; S^m\alpha)\rightarrow(S^nY, y_*; S^n\beta)$ by $\phi_m{}^n(S^mA)=S^n(\phi A)$, then the injection $\pi(\phi_m{}^n)\rightarrow\pi(\psi_m{}^n)$ may be embedded in an exact sequence, in which n remains fixed, which generalizes the familiar homotopy and cohomotopy sequences (note that $\pi(\phi)$ has a distinguished element, so that we may speak of an exact sequence). Defining $\pi_m(\phi)=\pi_s(\phi_m{}^0)$, $m\geqq 0$, $\pi_m(\phi)=\pi_s(\phi_0{}^{-m})$, $m<0$, we may proceed to the limit of the exact sequence mentioned to obtain an exact sequence

$$\cdots\rightarrow\pi_m(\phi)\rightarrow\pi_m(\psi)\rightarrow\pi_m(\psi, \phi)\rightarrow\pi_{m-1}(\phi)\rightarrow\cdots,$$

where the group $\pi_m(\psi, \phi)$ is defined in a manner suggested by the definition of the relative cohomotopy group. This sequence is used to study the S-extension and S-deformation problems, analogous, in the S-theory, to the usual extension and deformation problems. Let X be a CW-complex, A a subcomplex, $\alpha=\{X, A, x_*\}$, $\beta=\{Y, B, y_*\}$. Then the extension and deformation problems are equivalent to the S-extension and S-deformation problems "within the suspension range". *P. J. Hilton.*

Morita, Kiiti. Cohomotopy groups for fully normal spaces. Sci. Rep. Tokyo Bunrika Daigaku. Sect. A. **4**, 251–261 (1953).

E. H. Spanier has given a detailed investigation of Borsuk's cohomotopy groups [Ann. of Math. (2) **50**, 203–245 (1949); these Rev. **10**, 559] which is restricted to the case of compact spaces. By means of the results in some previous papers, the author is able to define the cohomotopy groups together with the induced homomorphisms and the coboundary operators for the category of countably fully normal spaces. It is then proved that they satisfy the Eilenberg-Steenrod axioms [Foundations of algebraic topology, Princeton, 1952; these Rev. **14**, 398] for cohomology theories except that the exactness axiom is only proved under a stronger condition that the spaces involved are fully normal.

The author also proves that the map excision theorem of A. D. Wallace for cohomology groups [Duke Math. J. **19**, 177–182 (1952); these Rev. **13**, 765] is also true for the cohomotopy groups of countably fully normal spaces; and it is indicated that the extension and reduction theorems of A. D. Wallace hold for the cohomotopy groups of fully normal spaces. At the end of the paper, the uniform cohomotopy groups are defined and briefly studied.

S. T. Hu (New Orleans, La.).

Rohlin, V. A. Intrinsic homologies. Doklady Akad. Nauk SSSR (N.S.) **89**, 789–792 (1953). (Russian)

The author continues his investigations on the "intrinsic homology groups" $\mathfrak{O}^k$ and $\mathfrak{N}^k$ of oriented, resp. non-oriented, smooth (not necessarily connected) manifolds, where ~ 0 (intr)[~ 0 mod 2 (intr)] means being boundary of an oriented [non-oriented] manifold [cf. same Doklady (N.S.) **81**, 355–357 (1951); **84**, 221–224 (1952); these Rev. **14**, 72, 573; related results in R. Thom, Colloque de topologie de Strasbourg, 1951, no. V, La Bibliothèque Nationale et Universitaire de Strasbourg, 1952; C. R. Acad. Sci. Paris **236**, 1733–1735 (1953); these Rev. **14**, 492, 1112]. No proofs are given. Results: If M^k bounds M^{k+1}, then, in the oriented case, all of Pontryagin's characteristic homology classes of M^k (except the Euler-Poincaré characteristic χ) bound in M^{k+1}, similarly for the Stiefel-Whitney homology classes in the non-oriented case. In particular the characteristic numbers, the Kronecker indices of the 0-dimensional elements of the characteristic homology ring vanish [this is essentially due to Pontryagin, Mat. Sbornik N.S. **21**(63), 233–284 (1947); these Rev. **9**, 243]; these numbers define therefore homomorphisms of $\mathfrak{O}^k$, resp. $\mathfrak{N}^k$, into the integers, resp. the integers mod 2. If $k=4m$, then the signature σ of the intersection matrix of the $2m$th homology group vanishes if M^k bounds, and so σ defines a homomorphism of $\mathfrak{O}^k$ into the integers; for $m=1$ this has been identified as $\frac{1}{3}$ of a certain Pontryagin number (second ref. above). The main result concerns the natural map h^k of $\mathfrak{O}^k$ into $\mathfrak{N}^k$. (I) The kernel of h^k is $2\mathfrak{O}^k$. (II) The $(k-1)$st Stiefel-Whitney homology class of M^k can be represented by an orientable submanifold A^{k-1} of M^k; the $(k-2)$nd Stiefel-Whitney homology class of the normal vector bundle of A^{k-1} in M^k can be represented by a submanifold B^{k-2} of A^{k-1}; the M^k lies in the image of h^k, if and only if A^{k-1} and B^{k-2} lie in the 0-classes of $\mathfrak{O}^{k-1}$, resp. $\mathfrak{N}^{k-2}$.

Real projective spaces P^k of even dimension $k=2r$ give non-zero elements of $\mathfrak{N}^k$: defining a characteristic number mod 2 by $\psi(M^k)=$ index of the kth power of the $(k-1)$st Stiefel-Whitney homology class, one has $\psi(P^{2r})\neq 0$. Complex projective spaces Q^{4r} of even complex dimension give an infinite cyclic subgroup of $\mathfrak{O}^{4r}$, since their signature σ is one.

$\mathfrak{O}^1$, $\mathfrak{O}^2$, $\mathfrak{O}^3$, $\mathfrak{N}^1$, $\mathfrak{N}^3$ are trivial; $\mathfrak{N}^2$ is cyclic of order two, $\mathfrak{O}^4$ infinite cyclic (generated by Q^4). Contrary to a statement in the second ref. above, $\mathfrak{N}^4$ is a direct sum of two groups of order two (generated by P^4 and Q^4). Every M^4 is equal (in $\mathfrak{N}^4$) to $\psi P^4+(\psi+\chi)Q^4$. The map h^4 is onto.

It should be noted that considerably stronger results on the structure of $\mathfrak{O}^k$ and $\mathfrak{N}^k$ have been announced by Thom (cf. ref. above).

H. Samelson (Princeton, N. J.).

Adem, José. Relations on iterated reduced powers. Proc. Nat. Acad. Sci. U. S. A. **39**, 636–638 (1953).

Let p be an odd prime and Z_p the field of integers modulo p. For any complex K, the cyclic reduced power operations [introduced by N. Steenrod, same Proc. **39**, 213–217, 217–223 (1953); these Rev. **14**, 1005, 1006] are linear maps

$$\mathcal{P}^s\colon\ H^q(K; Z_p)\rightarrow H^{q+2s(p-1)}(K; Z_p),\quad s=0, 1, 2, \cdots.$$

The main purpose of this paper is to give relations between various iterated cyclic reduced power operations, similar to the relations the author has previously obtained between iterated Steenrod squares [ibid. **38**, 720–726 (1952); these Rev. **14**, 306].

Let δ^*: $H^q(K; Z_p)\rightarrow H^{q+1}(K; Z_p)$ denote the so-called Bockstein coboundary operator. The author's basic results

assert that if $0\leqq r<sp$, then, (a) $\mathcal{P}^r\circ\mathcal{P}^s$ is equal to a linear combination of $\mathcal{P}^{r+s-i}\circ\mathcal{P}^i$ for $0\leqq i\leqq r/p$, and (b) $\mathcal{P}^r\circ\delta^*\circ\mathcal{P}^s$ is equal to a linear combination of $\delta^*\circ\mathcal{P}^{r+s-i}\circ\mathcal{P}^i$ and $\mathcal{P}^{r+s-j}\circ\delta^*\circ\mathcal{P}^j$ for $0\leqq i\leqq r/p$ and $0\leqq j\leqq (r-1)/p$.

From these basic results, the author derives several interesting consequences. Among these may be mentioned the following. (a) Any cyclic reduced power $\mathcal{P}^r$ can be expressed as a sum of iterated cyclic reduced powers with exponents powers of p. (b) The iterated powers of the type $\mathcal{P}^i\circ\mathcal{P}^j\circ\mathcal{P}^k\circ\cdots\circ\mathcal{P}^n$ with $i\geqq pj$, $j\geqq pk$, $\cdots$, and $i+j+k+\cdots+n=q$ form an additive base for all iterated powers $\mathcal{P}^a\circ\mathcal{P}^b\circ\cdots\circ\mathcal{P}^e$ such that $a+b+\cdots+e=q$. (c) Let $H^*(K)$ denote the integral cohomology ring of the complex K. $H^*(K)$ is said to be a truncated polynomial ring on u if it is generated by cup-product powers of u, and each of the integral cohomology groups is torsion-free. The height of u is the minimal integer n such that $u^n=0$. The author proves that if $H^*(K)$ is a non-trivial truncated polynomial ring generated by the cohomology class u, then the dimension of u is a power of 2; moreover, if the dimension of u is greater than 4, then the height of u is at most 3. (d) Various examples are known of fibre bundles in which the bundle space and fibres are spheres; e.g., the bundles having the complex and quaternionic projective spaces as base space, and the representation of S^{15} as a 7-sphere bundle over S^8, due to Hopf. The author proves that if any other such fibre bundles exist, then the bundle space and fibres must be spheres of dimensions $2^{k+1}-1$, 2^k-1 respectively, where k is an integer greater than 3.

Only very brief indications of the methods of proof are given. *W. S. Massey* (Providence, R. I.).

Bott, Raoul. On symmetric products and the Steenrod squares. Ann. of Math. (2) **57**, 579–590 (1953).

In this paper, the author obtains a definition of the Steenrod squaring operations [cf. Ann. of Math. (2) **48**, 290–320 (1947); these Rev. **9**, 154] directly from the exact sequences which are associated with the symmetric product of a space in the theory of P. A. Smith and M. Richardson [cf. appendix B of S. Lefschetz, Algebraic topology, Amer. Math. Soc. Colloq. Publ., v. 27, New York, 1942; these Rev. **4**, 84].

The main results may be stated as follows: Given a cell complex X, let $X*X$ denote the "symmetric product" which is obtained from $X\times X$ by identifying points which correspond to each other under the involution which interchanges the two "coordinates" of a point. Let p: $X\times X\to X*X$ denote the natural projection, and let Δ denote the "diagonal" in $X\times X$ and $X*X$. The map p induces homomorphisms p_m: $H^m(X*X,\Delta)\to H^m(X\times X,\Delta)$ for all m. The first part of the main result of this paper asserts that in cohomology modulo two, there exists a natural isomorphism ψ_n of the direct sum $G_n=H^n(X)+H^{n+1}(X)+\cdots+H^{2n-1}(X)$ onto the kernel of p_{2n}. The theory of Smith and Richardson gives rise to homomorphisms ϕ_m: $H^m(X\times X)\to H^m(X*X,\Delta)$, and the image of ϕ_m is contained in the kernel of p_m for all m. Let $u\in H^n(X)$; then $u\otimes u\in H^{2n}(X\times X)$, and $\phi_{2n}(u\otimes u)$ belongs to the kernel of p_m. Hence there exist unique elements $a_i\in H^{n+i}(X)$, $0\leqq i\leqq n-1$, such that

$$\phi_{2n}(u\otimes u)=\psi_n(a_0+a_1+\cdots+a_{n-1}).$$

The second part of the main results asserts that $a_i=\mathrm{Sq}^i(u)$, $0\leqq i\leqq n-1$, where Sq^i denotes the Steenrod square in cohomology modulo two.

The author acknowledges in a footnote that this theorem has been found independently by R. Thom [Colloque de topologie de Strasbourg, 1951, no. VII, Univ. de Strasbourg, 1952; these Rev. **14**, 491] and Wen-tsün Wu [ibid. no. IX; these Rev. **14**, 491]. Moreover, Thom and Wu have extended these results to the case of reduced pth powers for an arbitrary prime p, and have shown that the Steenrod squares and pth powers are uniquely determined by the requirement that they satisfy certain axioms.

W. S. Massey (Providence, R. I.).

Eilenberg, Samuel, and Mac Lane, Saunders. On the groups of $H(\Pi, n)$. I. Ann. of Math. (2) **58**, 55–106 (1953).

The calculation of the homology structure of the Eilenberg-MacLane complexes $K(\Pi, n)$ has acquired a crucial significance in the light of the Cartan-Serre technique of calculating homotopy groups of spaces by representing them as base-spaces in a fibring in which the fibre has the homotopy type of a geometrical realization of an Eilenberg-MacLane complex. The present paper is the first part of a systematic study of the complexes $K(\Pi, n)$; its particular object is to establish the equivalence of $K(\Pi, n)$ with a complex $A(\Pi, n)$, defined inductively from $A(\Pi, 0)=J(\Pi)$, the integral group ring of Π, by means of a 'bar' construction defined on the class of (commutative) R-complexes, a class which includes the complexes $K(\Pi, n)$, $A(\Pi, n)$.

The procedure adopted may be summarized as follows. First, FD-complexes are defined; these are sequences of abelian groups $K_0, K_1, \cdots, K_q, \cdots$, furnished with 'face' and 'degeneracy' operators $F_i{}^q$: $K_q\to K_{q-1}$, $D_i{}^q$: $K_q\to K_{q+1}$, subject to certain natural FD-commutation rules; such complexes generalize the complete semi-simplicial complexes of Eilenberg-Zilber [Ann. of Math. (2) **51**, 499–513 (1950); these Rev. **11**, 734]. The FD-complex K may be regarded in an obvious way as a chain-complex; the subcomplex, DK, of degenerate chains is defined and the factor-complex K_N, called the normalized complex of K, is such that the projection $K\to K_N$ is a chain-equivalence. A product $a_p\nabla b_q\in(K\times L)_{p+q}$, $a_p\in K_p$, $b_q\in L_q$, is defined on the cartesian product of two FD-complexes K, L, which generalizes the standard simplicial subdivision of the topological product of two simplexes. This product induces a product on the normalized complexes K_N, L_N to $(K\times L)_N$. An FD-complex is an R-complex if the K_q are rings with unit element 1_q, and if the $F_i{}^q$, $D_i{}^q$ are ring-homomorphisms satisfying $F_i{}^q1_q=1_{q-1}$, $D_i{}^q1_q=1_{q+1}$. The ∇-product induces in an R-complex the structure of a graded differential ring ('∂-ring'), which is skew-commutative if the rings K_q are commutative.

The 'bar' construction associates with a skew-commutative graded ∂-ring G a graded ∂-ring $B(G)$. The boundary operator in $B(G)$ is composed of a 'simplicial' boundary, ∂_s, and a residual boundary; $B(G)$ may be given the structure of a chain-complex (indeed, an FD-complex) with respect to which ∂_s is the boundary operator, so that $B_N(G)$ may be defined. It turns out that $B_N(G)$ is a skew-commutative graded ∂-ring with grading, boundary, and product induced by those in $B(G)$, and the projection $B(G)\to B_N(G)$ is a chain-equivalence. A suspension operator $G\to B(G)$ is based on a natural injection of G in $B(G)$, and raises by one the dimension of the homogeneous elements of G.

For an abelian group Π, $A(\Pi, n)$ is defined as

$$B_N{}^{n-1}B(A(\Pi, 0)).$$

We have a sequence of suspensions

$$A(\Pi, 0)\to A(\Pi, 1)\to A(\Pi, 2)\to\cdots,$$

and purely dimensional considerations show that the suspension $A(\Pi, n) \to A(\Pi, n+1)$ induces isomorphisms

$$H_q(A(\Pi, n)) \cong H_{q+1}(A(\Pi, n+1)), \quad 0<q<2n,$$
$$H^q(A(\Pi, n)) \cong H^{q+1}(A(\Pi, n+1)), \quad 0<q<2n,$$

with the usual weaker results in the critical dimensions.

Reverting to the complexes $K(\Pi, n)$, an isomorphism ω: $K_q(\Pi, n) \otimes K_q(\Pi, n+1) \cong K_{q+1}(\Pi, n+1)$ is established. This suggests a construction W which may be applied to any commutative R-complex (augmented by an admissible homomorphism of R into the integers). Then, as a special case, ω becomes an isomorphism

$$\omega: \quad W(K(\Pi, n)) \cong K(\Pi, n+1).$$

The equivalence of $A(\Pi, n)$ and $K(\Pi, n)$ is then established by exhibiting an isomorphism $A(\Pi, 1) \cong K(\Pi, 1)$ and the equivalence of $B(R)$ and $W(R)$; for it may readily be shown that if two skew-commutative augmented graded ∂-rings G and G', which are free as groups, are equivalent in the sense that there exists a map $f: G \to G'$ inducing the isomorphism of their homology rings, then there is an induced map $B(f): B(G) \to B(G')$ which induces an isomorphism of their homology rings.

On p. 59, l. 15, '$M(q)$' should be '$K(q)$', l. 20, 'u' should be 'u_q'; on p. 64, l. 36, 'K_{p+q+1}' should be '$(K \times L)_{p+q+1}$'; on p. 65, l. 12, 'b_p' should be 'b_q'; on p. 73, l. 19, '$(B)G$' should be '$B(G)$'; on p. 74, l. 8, 'α_i' should be 'α'; on p. 86, l. 28, '$=0$' should be '$=1$'; on p. 87, l. 4, 'm,' should be 'm_1'. The reference at the top of page 84 is either inadequate or superfluous. A notational innovation is the use of the Weierstrass elliptic function symbol $\wp(n)$ for $(-1)^n$. Notice of this innovation was given in a circular letter of the authors (sent from Nancago), and it may be said to stand up well to its first trial.

[Reviewer's note: A method for computing

$$H(K(\Pi, n); Z_2),$$

based on a fundamental theorem of A. Borel, is given by J.-P. Serre, Cohomologie mod 2 des complexes d'Eilenberg-MacLane, to appear in Comment. Math. Helv.]

P. J. Hilton (Cambridge, England).

GEOMETRY

Cobb, R. H. A symbolism for the geometry of the triangle. Math. Gaz. 37, 174–187 (1953).

***Boomstra, W. Triangles équilatères inscrits dans une conique donnée.** Anniversary Volume on Applied Mechanics dedicated to C. B. Biezeno, pp. 24–37. N. V. De Technische Uitgeverij H. Stam, Haarlem, 1953.

The problem of finding all equilateral triangles inscribed in a given conic K is solved as follows: the locus of the centers p of the circles C_p circumscribed to these triangles is shown to be again a conic. For each p the point q of $C_p \cap K$ other than the vertices of the triangle is given by an explicit formula; then the vertex of the triangle closest to q is calculated. The question of the constructibility of these triangles by ruler and compass is discussed. *H. Busemann.*

***Egerváry, Jenő. On the orthocentric coordinate system and on some of its applications.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 387–396. Akadémiai Kiadó, Budapest, 1952. (Hungarian. Russian summary)

A simplex is called orthocentric if its altitudes go through a point. The author proves among other things that a simplex of vertices $P_1, P_2, \cdots, P_n$ is orthocentric if and only if there exists n real numbers $\lambda_1, \lambda_2, \cdots, \lambda_n$ so that $\overline{P_iP_k}^2 = \lambda_i + \lambda_k$. He also investigates the Feuerbach spheres and Euler lines of these simplices. Finally he gives the following spatial analog of the theorem of Ptolemy: Let $P_1, P_2, P_3, P_4, P_5, P_6$ be a hexagon inscribed in a sphere. Assume that the lines P_1P_4, P_2P_5, P_3P_6 meet in a point. Then

$$(\overline{P_1P_4} \cdot \overline{P_2P_5} \cdot \overline{P_3P_6})^{2/3} = \overline{P_1P_2} \cdot \overline{P_4P_5} + \overline{P_2P_3} \cdot \overline{P_5P_6} + \overline{P_3P_4} \cdot \overline{P_6P_1}.$$

P. Erdős (South Bend, Ind.).

***Höhn, Walter. Winkel und Winkelsumme im n-dimensionalen euklidischen Simplex.** Thesis, Eidgenössische Technische Hochschule Zürich, 1953. 39 pp.

Given an n-dimensional simplex, spherical or Euclidean, let s_k denote the sum of the angles at its $\binom{n+1}{k}$ $(n-k)$-cells, each measured as a fraction of the whole angle at the $(n-k)$-flat considered; e.g., $s_0 = 1$, $s_1 = \frac{1}{2}(n+1)$. Poincaré [C. R. Acad. Sci. Paris 140, 113–117 (1905)] showed that $\sum_0^n (-1)^k s_k = \{1 + (-1)^n\}\Omega$, where Ω is the content of the simplex as a fraction of the whole space (so that $\Omega = 0$ in the Euclidean case). By integrating a product of "characteristic functions", the author obtains the inequality

$$\binom{[\frac{1}{2}(n+1)]}{k} + \binom{[\frac{1}{2}(n+2)]}{k} \leqq 2s_k \leqq \binom{n}{k}$$

and the equations

$$\sum_{k=0}^{p} (-1)^k \binom{n+1-k}{n+1-p} s_k = s_p,$$

which include Poincaré's formula as the case $p = n+1$ if we make the convention $s_{n+1} = \Omega$. These hold for all values of p from 1 to $n+1$, but a set of independent equations is given by the alternate values (including $n+1$ itself). The author had evidently not seen the work of D. M. Y. Sommerville [Proc. Roy. Soc. London. Ser. A. 115, 103–119 (1927), (3.52)], who gave a different set of the same number of equations. In the author's notation, these are

$$\sum_{k=r}^{n+1} (-1)^k \binom{k}{r} s_k = \sum_{k=n+1-r}^{n+1} (-1)^{n+1-k} \binom{k}{n+1-r} s_k \qquad (r = 0, 1, \cdots, [\tfrac{1}{2}n]).$$

The equivalence of the two sets of equations is easily verified for small values of n, but far from obvious in the general case.

H. S. M. Coxeter (Toronto, Ont.).

Cadwell, J. H. A property of linear cyclic transformations. Math. Gaz. 37, 85–89 (1953).

Let $x_1^0 = x_{n+1}^0, x_2^0, \cdots, x_n^0$ be the vertices of a polygon in E^k and define by induction new polygons $x_1^i, \cdots, x_n^i$ through $2x_r^{i+1} = x_r^i + x_{r+1}^i$ or $2x_r^{i+1} = x_{r-1}^i + x_{r+1}^i$. Each of these sequences of polygons is shown to converge and estimates for the rapidity of the convergence are given.

H. Busemann (Los Angeles, Calif.).

★Kárteszi, Ferenc. On spheres touching the sides of skew quadrangles. Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 613–618. Akadémiai Kiadó, Budapest, 1952. (Hungarian. Russian and English summaries)

The author considers the spheres tangent to the sides of a skew quadrilateral. He proves that the number of these spheres is in general 8. The inner and outer bisecting planes (i.e., the planes bisecting the angles of the skew quadrilateral) form two tetrahedra whose vertices are the centers of these spheres. The vertices of each of these tetrahedra are on the faces of the other. A sphere which is tangent to the sides of the skew quadrilateral is called an inscribed sphere if the points of tangency are interior points of the sides of the quadrilateral. The author proves that the number of these spheres is 0 or ∞. In fact, inscribed spheres exist if and only if the sum of the length of two opposite sides of the quadrilateral are equal. *P. Erdös* (South Bend, Ind.).

Palazzo, Elena, Sulla totalità delle ellissi aventi lo stesso asse maggiore e nuove costruzioni di quelle. Ricerca, Napoli 4, no. 1–2, 59–68 (1953).

Giovanardi, Mario. Sulla prospettiva di una sfera. Ricerca, Napoli 4, no. 1–2, 40–51 (1953).

Brauner, Heinrich. Kongruente Verlagerung kollinearer Räume in achsiale Lage. Monatsh. Math. 57, 75–87 (1953).

Among the ∞^{15} self-collineations of euclidean space, there are ∞^9 biaxial homographies β and ∞^6 congruence transformations κ, and the question may be asked as to whether the general collineation ϖ can be expressed as a product $\beta\kappa$ of collineations of the two special types in question. To investigate this problem, the author first observes that ϖ can be replaced, for the purposes of the problem, by any collineation $\varpi\kappa$, where κ is a displacement (Bewegung), and by a suitable choice of κ the displaced collineation is "normalized" so that it admits of representation in homogeneous rectangular cartesian coordinates by equations of the form

$$x'=\alpha y,\quad y'=\beta x,\quad z'=\gamma t,\quad t'=\delta z\quad (\alpha\beta\gamma\delta\neq 0),$$

where $t=0$ is the plane at infinity. It is shown then that this normalized collineation, and hence also ϖ itself, can be congruently transformed to a biaxial collineation if and only if $\alpha^2\beta^2=\gamma^2\delta^2$, the significance of this condition being that the original factorization problem is poristic. The author gives an elegant geometrical analysis of the problem and a geometric form for the critical condition. He shows also, for example, that if ϖ admits factorization $\beta\kappa$, where κ is a displacement (the alternative $\alpha\beta=\gamma\delta$), then it admits a simple infinity of such factorizations, and the axes of the corresponding displacements (Umwendungsachsen) generate a Plücker conoid. Further details and special cases are discussed. *J. G. Semple* (London).

Raffaele, Polemio. Geometrie fondamentali e geometrie non euclidee quadrimensionali sulla ipersfera di Riemann dell'S_5. Ricerca, Napoli 4, no. 1–2, 52–58 (1953).

Si preannunziano alcuni risultati relativi alle geometrie fondamentali e alle geometrie non euclidee quadrimensionali, nella loro rappresentazione sopra una ipersfera dell'S_5. *Sunto del autore.*

Freudenthal, Hans. Zur ebenen Oktavengeometrie. Nederl. Akad. Wetensch. Proc. Ser. A. 56 = Indagationes Math. 15, 195–200 (1953).

A class of hermitian matrices over the Cayley numbers is used to construct a plane geometry. In this notation a point and a line are incident if their Jordan product is zero. Using appropriate conventions he can define a determinant and a characteristic polynomial for such matrices. In this way various properties usually available only in associative systems are obtained. *Marshall Hall, Jr.* (Columbus, Ohio).

★Hua, Loo Keng. Fundamental theorem of the projective geometry on a line and geometry of matrices. Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 317–325. Akadémiai Kiadó, Budapest, 1952. (Hungarian and Russian summaries)

Let z range over the projective line coordinatized by a division ring K of characteristic not two. It is shown that every mapping of L onto itself which preserves harmonic ratios $(z_2-z_4)^{-1}(z_2-z_3)(z_1-z_3)^{-1}(z_1-z_4)=-1$ is a generalized projective transformation, $z\to z'=(az^\sigma+b)(cz^\sigma+d)^{-1}$, where σ is an automorphism or antiautomorphism of K. *Marshall Hall, Jr.* (Columbus, Ohio).

Andreoli, Giulio. Spazi algoritmici proiettivi (su algebre di Boole). Giorn. Mat. Battaglini (5) 1(81), 42–69 (1952).

Analog zu der projektiven Geometrie auf der Geraden als ein Calcül mit 2-reihigen Matrizen über den reellen Zahlen begründet Verf. eine Art projektive Geometrie in einer einparametrigen Mannigfaltigkeit S_1 als ein Calcül 2-reihiger Matrizen über einer Boole'schen Algebra (B.A.). Die Matrix $\begin{pmatrix} p & q \\ p^* & q^* \end{pmatrix}$, wobei p^* das zu p konjugierte Element der B.A. ist, vermittelt eine Transformation $y=px+qx^*$ und gleichzeitig $y^*=p^*x+q^*x^*$. Diese Matrizen bilden im gewöhnlichen Sinne ein Gruppoid. Ist speziell $q=p^*$, so ist die Zuordnung zwischen x und y, resp. x^* und y^*, involutorisch; diese Matrizen bilden eine gewöhnliche Gruppe mit $\begin{pmatrix} T & N \\ N & T \end{pmatrix}$ als Einselement, wo N die leere, T die totale Klasse der B.A. ist. Zwischen Paaren geordneter Elemente der B.A. lässt sich eine symmetrische, reflexive und transitive Äquivalenzrelation erklären $(x, y)\sim(\xi, \eta)$, wobei $\xi=px+p^*x^*$, $\eta=py+p^*y^*$ ist. Eine Klasse äquivalenter Paare heisst Punkt des S_1. Als Repräsentant einer Klasse lässt sich ein Paar wählen, mit festem 1. Element: $(x, y)\sim(\alpha, z)$. Eine projektive Abbildung $(\alpha, z)\to(\alpha, \zeta)$ wird definiert durch $\zeta=(r\alpha+s^*\alpha^*)z+(s\alpha+r^*\alpha^*)z^*$. Analog wie bei gewöhnlichen projektiven Abbildungen lassen sich Normalformen angeben und die Diskussion der Fixelemente durchführen. Eine projektive Abbildung in einer B.A. ist durch höchstens 2 entsprechende Punktepaare bestimmt, die jedoch nicht willkürlich sind und gegebenenfalls die Abbildung nicht immer eindeutig bestimmen. Abschliessend gibt Verf. Beispiele einer B.A. und der in ihr erklärten Involutionen mit Hilfe eines arithmetischen und eines geometrischen Modells. *R. Moufang* (Frankfurt a.M.).

★Lombardo-Radice, Lucio. Su alcuni modelli di geometrie proiettive piane finite. Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 370–373. Casa Editrice Perrella, Roma, 1953.

In the Euclidean plane with coordinates from the field with a prime number p of elements a polygon with p sides

is constructed which is regular in that (1) the lines from any one vertex to the rest all have different directions, (2) the diagonal joining vertices i and $i+(p-1)/2$ is parallel to the opposite side, and (3) the midpoint of a side has the same abscissa as the opposite vertex. It is suggested that a regular polygon may be a useful model in attempting to find new finite planes. *Marshall Hall, Jr.* (Columbus, Ohio).

Wyler, Oswald. Order in projective and in descriptive geometry. Compositio Math. 11, 60–70 (1953).

It is customary to treat the notion of order in projective and descriptive geometry separately. This paper presents a unified theory of order, valid in either geometry. Specifically, the author assumes, in addition to Hilbert's axioms of incidence, seven axioms of order (separation of pairs of lines in a pencil is the primitive relation). He then shows that, as a consequence of these assumptions, the geometry is either projective geometry or descriptive geometry, each with its usual order properties. *A. J. Hoffman.*

Convex Domains, Extremal Problems, Integral Geometry

Levi, F. W. Eine Ergänzung zum Hellyschen Satze. Arch. Math. 4, 222–224 (1953).

Let F be a family of convex subsets of euclidean n-space, and consider the two propositions (for $m=1, 2, \cdots$): (A_m) the complement of the sum of each m sets of F is connected; (B_m) the product of each m sets of F is non-empty. Two theorems are proved: (1) if the sets of F are all closed, then B_{n+1} (and, by the Helly theorem, B_m for every m) follows from B_n and A_{n+1}; (2) if the sets of F are all bounded and closed, then A_s is valid for every $s\leqq n$, and $n>1$. [Reviewer's note. Hilfssatz 2 of the paper, to the effect that a convex subset of the n-sphere, that does not contain a diametral pointpair, is contained in a small cap, is well-known. See, for example, the reviewer's Theory and application of distance geometry, Oxford, 1953, p. 194, Lemma 74.1; these Rev. 14, 1009.] *L. M. Blumenthal.*

***Sós, Vera. On curves and surfaces which are convex with respect to a point or direction.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 643–652. Akadémiai Kiadó, Budapest, 1952. (Hungarian. Russian and English summaries)

The present paper originated in an investigation of Rényi on schlicht functions [Publ. Math. Debrecen 1, 18–23 (1949); these Rev. 11, 92]. Let g be a Jordan curve with continuously turning tangent. The author determines all the points P such that every line through P intersects g in at most two points. Also the directions are determined such that any line parallel to this direction intersects g in at most two points. Both sets are constructed by the use of the inflexion tangents of g. Analogous problems in space are also discussed. *G. Szegö* (Stanford, Calif.).

Bieri, H. Ein (M, F)-Problem mit Nebenbedingung. Experientia 9, 207–209 (1953).

The problem is to find among the convex bodies of revolution of given length and surface area that one which possesses the greatest value of the integral of the mean curvature. The author conjectures that the extremal bodies are the cones. This is proved by restricting the admitted bodies to two well-defined subclasses. The general problem seems to involve difficulties. *L. Fejes Tóth* (Budapest).

Grotemeyer, K. P. Über die Verbiegung konvexer Flächen mit Rand. Math. Z. 58, 41–45 (1953).

Call cap a surface of class C''' homeomorphic to a disk with positive Gauss curvature and a boundary curve of class C'' which is, moreover, intersected at most once by the lines with a suitable fixed direction. Two intrinsically isometric caps are congruent if their boundary curves are congruent. This was proved by Pogorelov in a more complicated way for convex caps without any differentiability hypotheses. A cap with a plane boundary is rigid under infinitesimal (isometric) transformations which leave the boundary curve, or only its curvature, stationary. *H. Busemann.*

Green, John W. Length and area of a convex curve under affine transformation. Pacific J. Math. 3, 393–402 (1953).

The author considers in the plane the class $A(K)$ of all curves into which a given convex curve K can be affinely transformed and seeks the minimum of the isoperimetric ratio $r=u^2/f$, where u denotes the perimeter and f the area. A necessary and sufficient condition that K yield the minimum of r over the class $A(K)$ (then K is called an "extreme curve") is that the second Fourier coefficients of the supporting function of K be zero. As a consequence, each extreme curve with a continuous radius of curvature has at least six vertices. Finally, the author seeks the maximum of r over all extreme curves. He is not aware that the result, given by the equilateral triangles, is known [F. Behrend, Math. Ann. 113, 713–747 (1937)]. *L. A. Santaló.*

Gustin, William. An isoperimetric minimax. Pacific J. Math. 3, 403–405 (1953).

A direct and simple proof of the following theorem of Behrend, proposed anew by Green (see the preceding review): Let $r=u^2/f$ be the isoperimetric ratio of the plane convex curve K; then the maximum taken over all K of the minimum of r taken over all affine transforms of K is attained by the equilateral triangles. *L. A. Santaló.*

Hartman, S. Über die Abstände von Punkten $n\xi$ auf der Kreisperipherie. Ann. Soc. Polon. Math. 25 (1952), 110–114 (1953).

Let ξ be an irrational number, and let $P_1, \cdots, P_n$ be points on the circumference of a circle of unit circumference, obtained by proceeding by successive steps of arc-length ξ from a given point P_0. Now consider the points $P_0, P_1, \cdots, P_n$ in the order in which they lie on the circumference, and let m_n be the shortest arc between two neighbouring points and M_n the longest arc. The author proves that, for almost all ξ,

$$\liminf_{n\to\infty} nm_n=0, \quad \limsup_{n\to\infty} nm_n=1,$$

$$\liminf_{n\to\infty} nM_n=1, \quad \limsup_{n\to\infty} nM_n=\infty,$$

as was conjectured by Steinhaus. The proofs are not difficult, and the results are shown to hold for any ξ which has unbounded partial quotients in its continued fraction development. *H. Davenport* (London).

Santaló, L. A. Measure of sets of geodesics in a Riemannian space and applications to integral formulas in elliptic and hyperbolic spaces. Summa Brasil. Math. 3, 1–11 (1952).

In a Riemann space with line element $F=(g_{ij}X^iX^j)^{1/2}$, $p_i=\partial F/\partial x^i$, a density for geodesics is introduced by the absolute value of the exterior differential form

$$dg=\sum_{i=1}^{n}[dp_1dx^1\cdots dp_{i-1}dx^{i-1}dp_{i+1}dx^{i+1}\cdots dp_ndx^n].$$

Many formulas for E^n generalize immediately: the number of intersections of a geodesic with a hypersurface integrated over all geodesics is proportional to the area of the hypersurface. The length of the arc of a geodesic through a point p inside a domain D and intercepted by D, integrated over the geodesics through p, is proportional to the volume of D. Various integral-geometric formulae established by others (sometimes only in 2 or 3 dimensions) for spaces of constant curvature are shown to follow readily from the general formulae for Riemann spaces. *H. Busemann.*

Algebraic Geometry

Villa, Mario. Una cubica collegata ad un punto unito di una trasformazione puntuale. Atti Accad. Ligure 9 (1952), 165–175 (1953).

Let O be a united point of a point-transformation T of a plane into itself. The united points of the osculating quadratic transformations (other than O) determine a plane involution of triads of points. The coincidence curve, in general, is a rational plane cubic with a node at O, whose tangents are the self-corresponding rays of the projectivity determined by T among the directions through O. The author finds the equation of this cubic, and examines various special cases into which it degenerates. *J. A. Todd.*

***Amato, Vincenzo. Sulle curve algebriche a gruppo di monodromia totale.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. I, pp. 164–166. Casa Editrice Perrella, Roma, 1953.

Some brief expository remarks. *H. T. Muhly.*

Vesentini, Edoardo. Sul comportamento effettivo delle curve polari nei punti multipli. Ann. Mat. Pura Appl. (4) 34, 219–245 (1953).

Let C be an irreducible plane algebraic curve with a singularity at the origin $O=O_0$, and let it be assumed for simplicity that C has only one branch, of order $\nu\geqq 2$, at O. Let $(O_i)=(O_0, O_1, O_2, \cdots)$ be the simple consecutive sequence of points on C which begins with O, the points of this sequence arranging themselves, as is known, in alternate sets of free points and satellite clusters; and let $C'(P)$ be the first polar of a generic point P of the plane with respect to C. The main problem with which the present paper deals is that of discovering what can be said generally about the behavior of $C'(P)$ at O, and more particularly what can be said about the multiplicities of $C'(P)$ at point of the sequence (O_i). The author extends the investigation in many instances by allowing P to be any point, not merely the generic point, of the plane. The methods used are arithmetical, in the sense that, of the two principal ideas he exploits, one is a systematic examination of all possible pairs of Newton diagrams of curves C and $C'(P)$, this resulting, by easy extension, in rules determining all possible types of behavior of $C'(P)$ over the first group of free points and subsequent satellite cluster of C at O; while the second idea is that of applying successive plane transformations which would permit investigation of the behavior of $C'(P)$ at any subsequent group of free points and terminal satellite cluster by exactly the same type of Newton diagram analysis as was used for the first. The general problem is more complicated than has hitherto been supposed, examples which illustrate this having been given recently by B. Segre [same Ann. (4) 33, 5–48 (1952); these Rev. 14, 683]. Enriques' supposed solution (for generic P), which he embodied in his well-known law of alternation, seems to have been based on the erroneous assumption that the behavior of $C'(P)$ at the points of (O_i) depends solely on the formal character of the sequence (O_i) (its proximity relations), being therefore independent of any other possible projective peculiarity of this sequence or any other kind of property of C.

The results claimed by the author are indeed extensive, those relating to the case in which there is only one satellite cluster in the sequence (O_i) giving already much insight into the whole question. He claims, for example, to have formulated conditions that C and P must satisfy in order that the law of alternation should correctly describe the behavior of $C'(P)$; and these conditions, it is claimed, are satisfied when P is a generic point of the plane, and C is a generic curve of its order which has an "assigned" singularity at O. Other theorems delimit the extent of possible variations from the law in question. A feature of the paper that seems to call for comment is the nature of the author's mechanism for passing beyond each satellite cluster in turn. The plane transformations which he uses for this purpose, though extraordinarily convenient analytically, are not birational and certainly not compounded of dilatations; so that his interpretation of their effects as giving direct information about the behavior of $C'(P)$ at the points of (O_i) is open to question. If the transformations in question could be adequately replaced by successions of dilatations, the difficulty in question ought not to arise. *J. G. Semple.*

Campedelli, Luigi. Sulle singolarità delle curve algebriche. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 13, 234–238 (1952).

When the composition of a singularity of an irreducible plane algebraic curve C at a point O is taken to be defined progressively by a suitable sequence of plane quadratic transformations, an argument for the finiteness of the composition has sometimes been based on the erroneous assumption that the first polar ϕ of a general point of the plane necessarily passes through each explicit or implicit multiple point of C. In this paper the author proposes to replace this argument by the following. It is first to be shown (in a sense and by methods not here defined) that ϕ passes to "virtual" multiplicity $i-1$ through every i-fold point (explicit or implicit) of C; and it is then to be proved, by a method here described, that if C were to possess infinitely many multiple points consecutive to O, then ϕ would pass effectively (to a multiplicity not less than the virtual) through infinitely many of them. A similar procedure is suggested relative to isolated singular points of surfaces. The arguments of the paper, in the reviewer's opinion, seem to need some basic clarifications. *J. G. Semple.*

d'Orgeval, Bernard. Courbe de diramation de certains plans multiples. Bull. Soc. Roy. Sci. Liège 22, 188–194 (1953).

Derwidué, L. Sur des transformations crémoniennes de l'espace liées à un faisceau de surfaces du sixième ordre. Bull. Soc. Roy. Sci. Liège 22, 143–147 (1953).

Derwidué, L. Exemples de transformations crémoniennes de l'espace ayant des courbes fondamentales de seconde espèce présentant le cas de Montesano. Bull. Soc. Roy. Sci. Liège 22, 218–225 (1953).

Room, T. G. Corrigenda to the paper, Transformations depending on sets of associated points. Proc. Cambridge Philos. Soc. 49, 736 (1953).

See same Proc. 48, 383–391 (1952); these Rev. 13, 978.

Godeaux, Lucien. Sur certaines surfaces aux points desquelles sont associées des quadriques dégénérées. I, II. Bull. Soc. Roy. Sci. Liège 22, 139–142, 211–217 (1953).

Godeaux, Lucien. Remarque sur la surface de bigenre un d'Enriques. Bull. Soc. Roy. Sci. Liège 22, 125–130 (1953).

Spampinato, Nicolò. Le varietà dell'S_{11} complesso determinate da una superficie algebrica dell'S_3 complesso. Ricerca, Napoli 4, no. 1–2, 3–10 (1953).

Vaccaro, Giuseppe. Esame di singolarità superficiali. I. Superficie algebriche d'ordine n con punti $(n-2)$-pli inflessionali. Atti. Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 13, 373–378 (1952).

If a surface F^n in S_3 has an inflexional $(n-2)$-fold point O with harmonic polar plane ω, then F^n is invariant under the harmonic homology with vertex O and axial plane ω. Using this fact the author discusses further possible singular points of F^n, showing, for example, that if F^n has a further s-fold point P $(s\geqq 2)$, then OP is $(s-2)$-fold or $(s-1)$-fold for F^n according as P does or does not lie on ω. If F^n $(n>4)$ has two inflexional $(n-2)$-fold points O, O' whose harmonic planes ω, ω' contain O', O respectively, then OO' is a locus of inflexional $(n-4)$-fold points with fixed (degenerate) osculating cone; and under the same conditions, when $n=4$, OO' is a double line of F^4. When $n=3$, it appears that if a cubic surface F^3 has two (simple) inflexional points O, O', then OO', if it does not lie on F^3, meets it in a third inflexional point O''; if OO' lies on F^3, then either (i) each of O, O' lies on the harmonic polar plane of the other, or (ii) if this is not so, OO' is a locus of ∞^1 inflexional points and F^3 has two binodes lying on OO'. *J. G. Semple* (London).

***Gaeta, Federico. Ricerche intorno alle varietà matriciali ed ai loro ideali.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 326–328. Casa Editrice Perrella, Roma, 1953.

Let $\mathfrak{a}$ be the H-ideal generated by the minors of order p of a homogeneous $p\times q$ matrix $\mathfrak{A}_{pq}$ of forms in $n+1$ variables $(p\leqq q,\ q-p+1\leqq n)$, of rank $r=q-p+1$. Then $\mathfrak{a}$ defines a pure algebraic variety of dimension $d=n-(q-p+1)$. The author states two theorems. The first is that if $\mathfrak{A}_{p,p-1}$ is a subordinate matrix of $\mathfrak{A}_{pq}$ of maximum rank, then

$$\mathfrak{a}=(f_1, \cdots, f_r):(\mathfrak{A}_{p,p-1}), \quad (\mathfrak{A}_{p,p-1})=(f_1, \cdots, f_r):\mathfrak{a},$$

where $f_1, \cdots, f_r$ are forms (in fact certain bordered minors of $\mathfrak{A}_{pq}$). The second result is of similar type, but seems to contain misprints, and hence cannot be quoted. The geometrical content of the result is that the V_d represented by $\mathfrak{a}$ is the residual intersection of $n-d$ forms passing through a V_{n-2} of residual $p-1$. An alternative statement is that V_d is the d-dimensional characteristic system of the linear system adjoint to a certain multiple of a hyperplane section of a V_{n-2} of residual p. *J. A. Todd.*

***Roth, Leonard. Sulle V_3 algebriche che possiedono un sistema anticanonico.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 434–439. Casa Editrice Perrella, Roma, 1953.

The author constructs a number of threefolds on which the anticanonical system is effective, and shows that irregular threefolds with this property exist. *J. A. Todd.*

***Muracchini, Luigi. Sulle varietà del Veronese.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 412–416. Casa Editrice Perrella, Roma, 1953.

The author announces the following theorem. If a V_k has a regular sth osculating space at every point, and possesses $\infty^{(h+1)(k-h)}$ varieties V_h $(1\leqq h\leqq k-1)$ quasi-asymptotic $\sigma^{\tau}{}_{s-1,s+1}$ of maximum species $\tau=\binom{h+s}{s+1}$, then V_k is the Veronesean of forms of order s in S_k. For $h=1$ this is a theorem of Bompiani and Villa. The author gives a proof in the case $k=3$, $h=2$, $s=2$, and states that the general result can be proved by induction on h. *J. A. Todd.*

Muracchini, Luigi. Trasformazioni puntuali fra due spazi a configurazione caratteristica armonica. Boll. Un. Mat. Ital. (3) 8, 144–152 (1953).

The author studies transformations between two S_3 such that, at every regular pair of corresponding points there are six plane calottes of the second order which correspond. It is shown that there exist transformations of this type depending on 12 arbitrary functions of one variable.

J. A. Todd (Cambridge, England).

***Muracchini, Luigi. Trasformazioni puntuali e loro curve caratteristiche.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 417–424. Casa Editrice Perrella, Roma, 1953.

The author states some problems in the theory of analytic point transformations between two planes which he is investigating, outlines his proposed method of approach, and states what he hopes to establish. The ultimate aim is a classification of such transformations based on the properties of the so-called characteristic curves, defined as follows. Let $x=f(u,v)$, $y=\phi(u,v)$ be the equations of an analytic transformation between the (x,y) and (u,v) planes. To the lines of the (x,y) plane correspond the integral curves of an equation of the form

$$v''=av'^3+bv'^2+cv'+d,$$

where a, b, c, d are functions of u, v defined by f and ϕ. The characteristic curves in the (u,v) plane are the integral curves of the differential equation $av'^3+bv'^2+cv'+d=0$. The author's main idea seems to be to determine what transformations T give rise to a given system of characteristic curves. The functions a, b, c, d must satisfy certain conditions for T to exist, and the classification depends in

part on the rank of the matrix

$$\begin{Vmatrix} 3a & -2b & c & 0 \\ 0 & 3a & -2b & c \\ b & -2c & 3d & 0 \\ 0 & b & -2c & 3d \end{Vmatrix}.$$

J. A. Todd (Cambridge, England).

★**Vaona, Guido. Sulle trasformazioni puntuali di 2ª e 3ª specie fra piani.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 449–455. Casa Editrice Perrella, Roma, 1953.

Let T be a point-correspondence between two planes. At a general point, there are three characteristic lines (unless T is a homography, when every line is characteristic). T is of the first, second, or third kind according as there are three, two, or one distinct characteristic lines at the general point. Theorems: I. If T is of the first kind, the characteristic lines are inflexional of the second species if and only if the characteristic curves are lines. II. If T is of the third kind, the characteristic lines are in general inflexional of the third species, and are of the fourth species if and only if the characteristic curves are lines. III. If T is of the second kind, the double characteristic lines are in general inflexional of the second species, and are of the third species if and only if the double characteristic curves are lines. IV. If T is of the third kind, the characteristic projectivity induced between a general pair of characteristic lines approximates T to the third order. *J. A. Todd.*

Differential Geometry

Godeaux, Lucien. Sur les surfaces avant mêmes quadrilatères de Demoulin. II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **39**, 363–368 (1953).

★**Villa, Mario. Per una geometria proiettiva differenziale in grande delle trasformazioni puntuali.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. I, pp. 263–273. Casa Editrice Perrella, Roma, 1953.

Expository lecture. *J. A. Todd.*

van Bouchout, V. Ueber die Verbiegung einer Kongruenz mit invariantem mittleren Parameter. Simon Stevin **29**, 125–130 (1952).

La déformation dont il s'agit est la déformation à la Beltrami d'une congruence rectiligne (C), c'est-à-dire la déformation que l'on obtient en supposant les rayons de (C) issus des différents points d'une surface S et en déformant S, au sens de Gauss, les rayons de (C) restant invariablement liés aux plans tangents correspondants. Prenant, sur S, comme lignes coordonnées u les courbes enveloppées par les projections des rayons de (C) sur les plans tangents correspondants de S et comme courbes v leurs trajectoires orthogonales $[ds^2=Edu^2+Gdv^2]$, l'auteur étudie les cas de déformation avec invariance des paramètres moyens. Il est conduit à trois types de solutions, (C_{n1}), (C_{n2}), et (C_3). Les congruences (C_{n1}) et (C_{n2}) sont normales et existent pour une surface S quelconque. (C_{n1}) est la congruence des normales à S, et (C_{n2}), qui est d'ailleurs la congruence normale la plus générale de l'espace, s'obtient en prenant les droites issues des différents points de S formant avec leurs projections sur les plans tangents (tangentes aux courbes u) l'angle ω tel que $\cos\omega=U/E^{1/2}$, U étant une fonction arbitraire de u. Les congruences (C_3) n'existent que pour les surfaces S applicables sur les surfaces de révolution, et on les obtient en traçant dans les plans normaux à S le long des tangentes aux déformées des parallèles (courbes u) des droites faisant avec ces tangentes un angle ω fonction de v seul. Ces congruences (C_3) jouissent de deux autres propriétés d'invariance remarquables: Sur chaque rayon le produit des distances des deux points limites au point correspondant M de S reste constant lorsque S se déforme, et la même invariance a lieu pour le produit des distances de M aux deux foyers. D'autre part, pour chaque configuration particulière de S, les surfaces reglées de (C_3) issues des asymptotiques de S admettent ces mêmes asymptotiques pour lignes de striction. *P. Vincensini* (Marseille).

Grotemeyer, Karl-Peter. Zur infinitesimalen und endlichen Verbiegung von Halbeiflächen. Arch. Math. **4**, 52–60 (1953).

Given a three-times continuously differentiable surface with an everywhere positive Gaussian curvature and which is bounded by a closed plane geodesic curve. The author proves that it is congruent to its image under any finite or infinitesimal isometry that does not change the curvature or torsion of its boundary curve. The following references might be added: Stoker, Comm. Pure Appl. Math. **3**, 231–257 (1950) [these Rev. **12**, 631]; A. V. Pogorelov, Izgibanie vypuklyh poverhnosteĭ, Gostehizdat, Moscow-Leningrad, 1951 [these Rev. **14**, 400]. *P. Scherk.*

Poznyak, È. G. Infinitesimal deformations of troughs. Mat. Sbornik N.S. **32**(74), 681–692 (1953). (Russian)

This is a detailed exposition of the results announced previously [Doklady Akad. Nauk SSSR (N.S.) **78**, 205–207 (1951); these Rev. **12**, 857]. The author observes that no assumption of analyticity is made. *L. C. Young.*

Wunderlich, Walter. Zur Differenzengeometrie der Flächen konstanter negativer Krümmung. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. **160**, 39–77 (1951).

A skew lattice L is a topological image in space of the edges and vertices of the plane square lattice. Each edge of L is a segment a of length $s(a)$. Thus the image of each fundamental square is a skew quadrangle. The four edges of L adjoining each vertex are assumed to be coplanar. Thus L has a well-defined tangent plane and normal at each vertex and a spherical image can be introduced. The torsion of the edge a is defined by means of $\tau(a)=\sin\sigma(a)/s(a)$ where $\sigma(a)$ is the angle between the tangent planes at the endpoints of a. Let $s(a)$ be constant (rhomboid lattice L_0). Then $|\sigma(a)|$ and $|\tau(a)|$ are also constant. The author proves an analogue for L_0 of Hazzidakis' theorem on the area of a quadrangle bounded by asymptotic lines on a surface S of constant negative curvature [J. Reine Angew. Math. **88**, 68–73 (1880)]. Parallel and center lattices of L_0 are introduced. L_0 can be realized as a stable net of threads held tight by a suitable system of forces and also by means of a mechanism composed of special congruent Bennett isograms [Proc. London Math. Soc. (2) **13**, 151–173 (1914)].

L is a skew parallelogram lattice L_1 if the images in L_1 of parallel edges of the plane lattice have the same lengths and if $|\tau(a)|$ remains constant. A suitable limit process transforms L_1 into the net of the asymptotic lines of a surface S. By means of the L_0's and L_1's finite analogues of both

Lie's and Bäcklund's transformations of the surfaces S are constructed. *P. Scherk* (Saskatoon, Sask.).

Blanuša, Danilo. Eine isometrische und singularitätenfreie Einbettung des n-dimensionalen hyperbolischen Raumes im Hilbertschen Raum. Monatsh. Math. **57**, 102–108 (1953).

It is shown that an n-dimensional hyperbolic space can be imbedded isometrically, one-to-one, and without singularities in Hilbert space. Bieberbach had proved this result for $n=2$ by a different method, for which it is not clear whether it can be generalized to $n>2$. *H. Busemann.*

Bompiani, Enrico. Intorno alle rappresentazioni degli spazi a curvatura costante sullo spazio euclideo. Atti Accad. Ligure **9** (1952), 99–105 (1953).

The Killing-Darboux mappings of the non-Euclidean spaces on ordinary 3-dimensional spaces are obtained in a more geometrical form. Elliptic space E_3 is first treated as a projective space in which the absolute can always be given in homogeneous coordinates in the form

$$\Omega(x)\equiv x_0^2+x_1^2+x_2^2+x_3^2=0.$$

For points not on the absolute, coordinates of Weierstrass are taken, normalized so that $\Omega(x)=1$.

The author interprets the numbers $X=x_1/x_0$, $Y=x_2/x_0$, $Z=x_3/x_0$ as Cartesian orthogonal coordinates in a Euclidean space S_3 in which the absolute of E_3 becomes the complex sphere $X^2+Y^2+Z^2=-1$, and he shows how this leads naturally to a mapping between the Cartesian coordinates (ξ_1, ξ_2, ξ_3) of S_3 and the Weierstrassian coordinates (x_0, x_1, x_2, x_3) of E_3. There are two points of S_3 corresponding to a point of E_3, these two points are mutually inverse with respect to the complex sphere $\Omega(x)=0$. The author employs the same idea for the mapping of hyperbolic space on E_3, and proceeds to some further properties of the mappings. *E. T. Davies* (Southampton).

Blaschke, W. Zur topologischen Differentialgeometrie. J. Reine Angew. Math. **191**, 153–157 (1953).

A new development is given of the local invariant theory of a three-web of curves, using Cartan's exterior differential calculus [for terminology cf. Blaschke and Bol, Geometrie der Gewebe, Springer, Berlin, 1938]. From this set-up the following theorem of J. Dubourdieu is easily derived. Given a four-web of surfaces in space. If the curves of intersection of the surfaces of three families by the surfaces of other families form hexagonal webs, the same is true of the three-webs on the surfaces of the fourth family. *S. Chern.*

Berger, Marcel. Sur les groupes d'holonomie des variétés riemanniennes. C. R. Acad. Sci. Paris **237**, 472–473 (1953).

Whenever the restricted homogeneous holonomy group of a non-symmetric Riemannian space is irreducible and of the second class (according to Cartan) it is simple or simple by T^1 [cf. A. Borel and A. Lichnerowicz, same C. R. **234**, 1835–1837 (1952); these Rev. **13**, 986]. *A. Nijenhuis.*

Flanders, Harley. A method of general linear frames in Riemannian geometry. I. Pacific J. Math. **3**, 551–565 (1953).

This paper develops the basic concepts of Riemannian geometry, such as parallelism, curvature tensor, etc., by using all linear frames. It therefore includes as special cases both the standard formulation of absolute differential calculus and the method of moving orthonormal frames. The treatment is lucid; formulas are generally simplified by using matrix notation. It is felt that one of the possible applications of this formalism would be to the geometry of spinors in a Riemannian space. *S. Chern* (Chicago, Ill.).

Norden, A. P. On intrinsic geometries of surfaces of a projective space. Trudy Sem. Vektor. Tenzor. Analizu **6**, 125–224 (1948). (Russian)

The present paper comprises the first two parts of a complete and detailed study of projective differential geometry. The first part deals with the differential geometry of an affinely connected space of two dimensions—without torsion—and the basic geometric tool is parallel displacement. Thus equiaffine geometry is defined as one in which there exists a tensor L_{ij} such that $\sigma=L_{ij}u^iv^j$ is invariant under parallel displacement. (In two dimensions this is equivalent to preserving the area of any parallelogram.) From this it follows that the symmetry of the Ricci tensor is necessary and sufficient for the space to be equiaffine. A Weyl geometry may be characterized by the existence of two independent absolutely parallel direction fields. Quasi-euclidean geometry has a skew symmetric Ricci-tensor. Projectively euclidean two-dimensional space is characterized by $\nabla^k(2R_{ki}+R_{ik})=0$, where R_{ik} is the Ricci tensor. The remainder of this part of the paper is devoted to related connections, the relation being defined by some common geometric property. The results are well known for a general space, but the method of obtaining them is particularly simple, essentially because in a two-dimensional space any skew symmetric tensor of order two has the form $a\epsilon_{ij}$ where a is a scalar. The second part is devoted primarily to the study of three-dimensional projective spaces but the method is applicable to spaces of higher dimensions. The results are mostly well known and the method, that of normalizing a subspace, has been developed by Norden and his coworkers over the past few years. Starting with an $(n+1)$-dimensional projective space, a hypersurface is given by point coordinates $x^\alpha(u)$ or dually by "line" coordinates $\xi_\alpha(u)$, $\alpha=1, 2, \cdots, n+2$, the coordinates being homogeneous, and u^i, $i=1, 2, \cdots, n$, being curvilinear coordinates in the hypersurface. A moving polyhedral is then constructed by choosing n points $y_k^\alpha=\partial_k x^\alpha-l_k x^\alpha$ in the tangent hyperplane and a point X^α on a line through x not in this tangent plane. Then a point connection is defined by $\partial_j y_i^\alpha=a^k{}_{ij}y_k^\alpha+p_{ij}x^\alpha+b_{ij}X^\alpha$ and of course dually by $\partial_j\eta_{\alpha i}=\alpha^k{}_{ij}\eta_{\alpha k}+\pi_{ij}\xi_\alpha+\beta_{ij}\Xi_\alpha$. The various types of normalization are then examined in terms of these connections. Probably the most interesting result deals with the various resulting geometries and it is shown, that of the six types considered, if the normalization belongs to any two of them, then it belongs to all six. *M. S. Knebelman.*

***Norden, A. P. On an interpretation of the complex affine plane.** Sto dvadcat' pyat' let neevklidovoĭ geometrii Lobačevskogo, 1826–1951 [One hundred and twenty-five years of the non-Euclidean geometry of Lobačevskiĭ, 1826–1951], pp. 187–194. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1952. 7.60 rubles.

Considered is a four-space B_4 whose underlying group is that subgroup of the affine group which preserves two imaginary conjugate lines at infinity. Such a space is called biaffine. The totality of all directions at a point is a projective space P_3 in which two (absolute) imaginary lines are fixed; such a space is called biaxial and has been in-

vestigated by the author in Mat. Sbornik N.S. **24**(66), 429–455 (1949) [these Rev. **11**, 135]. In such a P_3 there are lines which intersect the absolute lines; they are called singular. Correspondingly, through every point of B_4 pass ∞^2 singular planes; such a plane contains two "conjugate" vectors. The group of biaffine motions (transformations of the biaffine group) is isomorphic with the group of affine transformations of the complex plane A_2. The biaxial angular metric in P_3 is given by a tensor G_i^j for which $G_k^iG_j^k=-\delta_j^i$, $G_k^k=0$, and the correspondence of vectors in B_4 is characterized by $\bar{a}^i=\mathfrak{G}_k^ia^k$. If a mapping of B_4 on itself which preserves G_i^j is called biconformal, then, in terms of an appropriate coordinate system, such a biconformal transformation $y^i=y^i(x^1, x^2, x^3, x^4)$ satisfies the equation $y_i^p\mathfrak{G}_p^j=y_p^j\mathfrak{G}_i^p$ and can be written $y_1^1=y_2^2$, $y_2^1=-y_1^2$, $y_1^3=y_2^4$, $y_2^3=-y_1^4$, $y_3^1=y_4^2$, $y_4^1=-y_3^2$, $y_3^3=y_4^4$, $y_4^3=-y_3^4$ where $y_j^i=\partial y^i/\partial x^j$, i, j, k, $p=1, 2, 3, 4$. Hence the functions of A_2

$$y^1+iy^2=\varphi(x^1+ix^2, x^3+ix^4)$$
$$y^3+iy^4=\psi(x^1+ix^2, x^3+ix^4)$$

are analytic.

An outline is given of the theory of curves and of surfaces X_2 in B_4. The X_2 are singular if their tangent planes are singular; they correspond to analytical curves of A_2. The basic formula for non-singular surfaces can be written in the form

$$\nabla_\beta x_\alpha=\partial_\beta x_\alpha-\mathfrak{G}_{\beta\alpha}^\gamma x_\gamma=b_{\beta\alpha}^\gamma\bar{x}_\gamma,\quad \alpha, \beta, \gamma=1, 2,$$

where $x_\alpha=\partial x/\partial u^\alpha$, $\bar{x}_\alpha=\partial\bar{x}/\partial u^\alpha$, x the radius vector, $\bar{x}$ its conjugate. The conditions $b_{\beta\alpha}^\gamma$ are such that the curvature tensor belonging to $\Gamma^\gamma{}_{\alpha\beta}=G^\gamma{}_{\alpha\beta}\pm ib^\gamma{}_{\alpha\beta}$ is zero. When for an $X_2 b^\gamma_{\alpha\beta}=0$, then its tangent planes are pseudo-parallel, that is, the angle of their lines at infinity is zero.

D. J. Struik (Cambridge, Mass.).

★Širokov, A. P. On the problem of A-spaces. Sto dvadcat' pyat' let neevklidovoĭ geometrii Lobačevskogo, 1826–1951 [One hundred and twenty-five years of the non-Euclidean geometry of Lobačevskiĭ, 1826–1951], pp. 195–200. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1952. 7.60 rubles.

In an affine vector space of $2n$-dimensions we can associate two n-vectors $V^{i_1\cdots i_n}$ and $V_{i_1\cdots i_n}$ by means of the equations

$$n!V^{i_1\cdots i_n}=\epsilon^{i_1\cdots i_n\alpha_1\cdots\alpha_n}V_{\alpha_1\cdots\alpha_n},$$
$$n!V_{i_1\cdots i_n}=\epsilon_{\alpha_1\cdots\alpha_n i_1\cdots i_n}V^{\alpha_1\cdots\alpha_n}$$

when the ϵ are $2n$-vectors for which $\epsilon^{i_1\cdots i_{2n}}\epsilon_{k_1\cdots k_{2n}}=\delta_{k_1\cdots k_{2n}}^{i_1\cdots i_{2n}}$. In the projective P_{2n-1} of the pseudo-vectors two n-vectors V and W which have no vectors in common, determine two non-intersecting $(n-1)$-spaces and with it a point involution. To this involution belongs the tensor

$$g_i{}^j=V^{ja_2\cdots a_n}W_{ia_2\cdots a_n}-(-1)^nW^{ja_2\cdots a_n}V_{ia_2\cdots a_n}.$$

For $n=2$ we have the biaxial case of Norden [see the preceding review].

If we now take the A_{2n} with symmetrical affine connection $(\Gamma^\alpha{}_{\beta\gamma})$ in which two simple n-vectors V and W are given for which

$$\nabla_\alpha V^{\gamma_1\cdots\gamma_n}=\lambda_\alpha V^{\gamma_1\cdots\gamma_n},\quad \nabla_\alpha W^{\gamma_1\cdots\gamma_n}=\mu_\alpha W^{\gamma_1\cdots\gamma_n},$$

then the n-spaces of V and W form two families of n-dimensional surfaces. Then $\nabla_\alpha g_\beta{}^\gamma=0$. Two cases are now discussed; in the first case V and W are real, in the second V and W are complex conjugate. In the first case we can introduce a special set of n coordinate lines on the surfaces V and n coordinate lines on the surfaces W such that in the new coordinates $x^1\cdots x^n$, $x^{n+1}\cdots x^{2n}$

$$\partial_k g_{r,n+s}-\partial_r g_{k,n+s}=0,\quad \partial_{n+k}g_{r,n+s}-\partial_{n+s}g_{r,n+k}=0,$$
$$k, r=1, \cdots, n,$$

is the condition that the tensor $s_{\alpha\beta}$ is covariant constant. Here the tensor $s_{\alpha\beta}$ has the property that $a_{\alpha\beta}=g_\alpha{}^\epsilon s_{\epsilon\beta}$ is anti-symmetric, $a_{\alpha\epsilon}a_\beta{}^\epsilon=-s_{\alpha\beta}$.

These spaces are related to the stratified spaces of Raševskiĭ [see the paper reviewed below]. By taking $x^{i'}=x^i+x^{n+i}$, $x^{n+i'}=x^i-x^{n+i}$ we can associate to every point x^α (the $'$'s are omitted) and vector v^α a point X^k and vector V^k of an n-dimensional dual space

$$X^k=x^k+ex^{n+k},\quad V^k=v^k+ev^{n+k},\quad e^2=+1,$$

with a connection $\Gamma_{ij}^k=\Gamma_{ij}^k+e\Gamma_{n+i,n+j}^{n+k}$. This space is a dual unitary space with metrical tensor

$$A_{k\bar{r}}=s_{kr}+ea_{kr}=s_{kr}+es_{n+k,r}$$

and no torsion.

In the second case, by a new choice of coordinates x^α, and taking

$$X^k=x^k+ix^{n+k},\quad V^k=v^k+iv^{n+k},\quad '\Gamma_{ij}^k=\Gamma_{ij}^k-i\Gamma_{n+i,n+j}^{n+k}$$

we obtain a unitary space of Schouten with a hermitian metrical tensor $A_{k\bar{r}}=s_{kr}-ia_{kr}$. The tensor $s_{\alpha\beta}$ is covariant constant if

$$\partial_{[j}s_{m]k}-\partial_{[n+j}s_{n+m]k}=0,\quad \partial_{[j}s_{n+m]k}+\partial_{[n+j}s_{m]k}=0,$$

which points to the absence of torsion.

The relation between stratifiable and unitary spaces without torsion was pointed out by B. A. Rozenfeld [Trudy Sem. Vektor. Tenzor. Analizu **7**, 260–275 (1949); these Rev. **12**, 359]. It is stressed that the whole theory dates back in principle to the theory of A-spaces given by P. A. Širokov [Bull. Soc. Phys.-Math. Kazan (2) **25**, 86–114 (1925)].

D. J. Struik (Cambridge, Mass.).

Raševskiĭ, P. K. The scalar field in a stratified space. Trudy Sem. Vektor. Tenzor. Analizu **6**, 225–248 (1948). (Russian)

The notion of stratified spaces has been thoroughly developed by the Moscow seminar during the past decade with many interesting and significant results. The present paper is too highly specialized to be of much significance. Given a space of $2n$ dimensions with coordinates $x^1, x^2, \cdots, x^n$; $u^{\hat{1}}, u^{\hat{2}}, \cdots, u^{\hat{n}}$ the stratifying spaces are then $x^i=c^i$ and $u^{\hat{i}}=c^{\hat{i}}$. It is assumed that a scalar function $U(x, u)$ is given such that $|\partial^2U/\partial x^i\partial u^{\hat{j}}|\neq 0$ which is taken as the metric tensor in the space. Since the above condition is not invariant under a general coordinate transformation, the problem is studied under transformations of the x's and u's separately. After calculating the various sets of Christoffel symbols one can see that the stratifying spaces are totally geodesic, admit absolute parallelism and have null length. The last half of the paper is devoted to proving a replacement theorem in this special space, a theorem that has been established for general spaces.

M. S. Knebelman.

Freidina, M. G. Dual systems allowing a group of motions. Trudy Sem. Vektor. Tenzor. Analizu **6**, 420–443 (1948). (Russian)

A dual system consists of linear elements in a plane defined by $(x, y, z=dy/dx)$ in which two metrics are prescribed, viz: $ds_1=A_1dx+B_1dz$ and $ds_2=A_2dx+B_2dz$, the A's and B's being functions of (x, y, z) and dy missing since $dy=zdx$. This is a special case of a nonholonomic space in which $ds_\mu=\omega_i{}^\mu dx^i$, $\mu=1, 2$, $\omega_i{}^0dx^i=0$, the ω's being functions of (x^1, x^2, x^3). The

main purpose of the paper is to show that such a space admits at most a one-parameter group of motions and to construct such a space, that is, to give the explicit form of the ω's in a canonical coordinate system.

M. S. Knebelman (Pullman, Wash.).

Castoldi, Luigi. Attorno alla teoria delle connessioni.— Deduzione autonoma del carattere tensoriale dei sistemi quadrupli di curvatura. Boll. Un. Mat. Ital. (3) **8**, 127–130 (1953).

The author points out that the tensor character of the 4-index symbols formed from the connection coefficient and usually called the Riemann-Christoffel tensor is usually proved by the formula for the interchange of the order of the second tensor derivatives of an arbitrary vector. In this note he obtains the same result by a procedure which is independent of any extraneous element such as an arbitrary vector. The author bases his proof on the observation that if the coefficients of connection in a particular system of reference are identically zero, the corresponding 4-index symbols are also zero, not only in that system of reference, but in every other system also.

E. T. Davies.

Guggenheimer, Heinrich. Geometria pseudo-kähleriana. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **14**, 220–222 (1953).

A pseudo-kählerian manifold is a differentiable manifold V_{2n} satisfying the conditions: (i) on V_{2n} there exists everywhere a differentiable tensor field a_j^i satisfying the conditions

$$a_j^i a_k^j = -\delta_k^i; \quad \left(\frac{\partial a_p^i}{\partial x^s} - \frac{\partial a_s^i}{\partial x^p}\right) a_k^p = \left(\frac{\partial a_p^i}{\partial x^k} - \frac{\partial a_k^i}{\partial x^p}\right) a_s^p;$$

(ii) on V_{2n} there exists a closed 2-form $e = e_{ij} da^i da^j$ such that $e_{pq} a_i^p a_j^q = e_{ij}$. The author proves that on a pseudo-kählerian manifold there exists, locally, a function U such that

$$e = \frac{\partial^2 U}{\partial x^i \partial x^j} a_k^j da^i da^k.$$

W. V. D. Hodge (Cambridge, England).

Fieschi, R., and Fumi, F. G. High-order matter tensors in symmetrical systems. Nuovo Cimento (9) **10**, 865–882 (1953). See the note on p. 1139.

It is well known that the thirty-two groups of a crystal are determined by twenty-one transformations in space. From the principle of Neumann (the symmetry of the crystal determines the group of the crystal and, hence, the symmetry of its physical properties), various authors have determined the various tensors through the fourth order associated with a given crystal [J. A. Schouten, Tensor analysis for physicists, Oxford, 1951, pp. 152–169; these Rev. **13**, 493]. For problems of finite strain, spontaneous electric or magnetic polarization, it is necessary to know the tensors of the fifth and sixth orders associated with a crystal. In the present paper, the authors determine these tensors of the polar and axial types (that is, whose components do not or do change sign under the coordinate transformation of reflection). This is done by direct inspection for those crystals which possess one of the nineteen groups of cartesian symmetry (all crystals except those of the trigonal and hexagonal types). For the trigonal and hexagonal types of crystals, the authors use group theory [see F. Murnaghan, Finite deformations of an elastic solid, Wiley, New York, 1951, Chap. 5; these Rev. **13**, 600]. A table of results completes the paper.

N. Coburn (Ann Arbor, Mich.).

NUMERICAL AND GRAPHICAL METHODS

***Andreev, P. P. Matematičeskie tablicy.** [**Mathematical tables.**] Gosudarstv. Statist. Izdat., Moscow, 1952. 471 pp. 7.75 rubles.

The first 333 pages of this work tabulate the functions n^2, n^3, $n^{1/2}$, $(10n)^{1/2}$, $n^{1/3}$, $n^{-1/2}$ for $n=1(1)10000$. Values are given to 6 significant places only. The second part of the tables is devoted to a dozen small tables of reciprocals, logarithms, constants, annuity functions, probability integral, factorials and binomial coefficients. The printing is clear and not too irregular. The volume is apparently intended for students of statistics.

D. H. Lehmer.

***Wijdenes, P. Noordhoff's wiskundige tafels in 5 decimalen.** [**Noordhoff's mathematical tables to 5 decimal places.**] P. Noordhoff, Groningen, 1953. viii+269 pp. 8.75 florins.

Besides the usual five-figure tables one expects in handbooks of tables, this work contains an unusually good table of natural trigonometric functions, a factor table and list of primes to 11200, and a few small tables of higher functions including the exponential integral and Bessel functions. The printing is excellent with explanations in Dutch, Malayan, French, English, German, and Spanish.

D. H. Lehmer (Los Angeles, Calif.).

***Romig, Harry G. 50–100 binomial tables.** John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London, 1953. xxvii+172 pp. $4.00.

These useful tables give values of

$$b(x, n, p) = p^x \binom{n}{x} (1-p)^{n-x}$$

and the partial sum

$$B(x, n, p) = \sum_{\nu=0}^{x} b(\nu, n, p)$$

for $p=.01(.01).5$, $n=50(5)100$ and for such integer values of x as make $b(x, n, p)$ greater than $5\cdot 10^{-8}$. Values are given to 6 decimals the last being doubtful. Thus for $n=95$, $p=.31$ the x's range from 9 to 53. These tables extend more elaborate tables [Tables of the binomial probability distribution, Nat. Bur. Standards, Appl. Math. Ser., no. 6, Washington, D. C., 1950; these Rev. **11**, 692] for which $n=2(1)49$. The present tables were computed in 1947 by the Bell Laboratories Computer Model 5. The introduction gives data on the problem of interpolation, always bothersome in tables of this kind, and supplies examples of the use of the table for solving sampling problems. The language used is statistical rather than mathematical; thus we find: "The binomial is not continuous, but consists of $n+1$ finite terms". For many problems involving fairly good sized samples these tables will be extremely helpful.

D. H. Lehmer.

***Rosser, J. Barkley, and Walker, R. J. Properties and tables of generalized rocket functions for use in the theory of rockets with a constant slow spin.** Cornell University, Ithaca, N. Y., 1953. v+114 pp.

In the discussion of the motion during burning of a rocket with constant slow spin in their book "Mathematical theory of rocket flight" [McGraw-Hill, New York, 1947; these Rev. **9**, 108], J. B. Rosser, R. R. Newton and G. L.

Gross were led to consider two integrals which were there estimated for certain ranges of the parameters appearing in them. These integrals are now evaluated in the general case. The authors begin by solving the equations of motion of a rocket with constant slow spin subject to malalinements of constant modulus of the fins, jet and principal axes of inertia. The solution contains various single and double integrals of an oscillatory nature. In three appendices these functions are studied and expressed in forms, such as asymptotic series, suitable for calculation. The analysis follows the lines of J. B. Rosser's book "Theory and application of $\int_0^z e^{-x^2}dx$ and $\int_0^z e^{-p^2y^2}dy\int_0^y e^{-x^2}dx$" [Mapleton House, Brooklyn, N. Y., 1948; these Rev. **10**, 267]. In Table Ia the functions $\frac{1}{2}[\{\mathrm{Rr}(x)\}^2+\{\mathrm{Ri}(x)\}^2]$ and $\int_0^x \mathrm{Rr}(y)dy$ (see review of Rosser's book just quoted for notation) are tabulated to five decimals for $x=0.0(0.1)5.0$. Table I contains values of the real and imaginary parts of the functions

$$\mathrm{Rre}(\alpha, x)=\int_x^\infty \mathrm{Rr}(w)e^{i\alpha(w-x)}dw,$$

$$\mathrm{Rie}(\alpha, x)=\int_x^\infty \mathrm{Ri}(w)e^{i\alpha(w-x)}dw,$$

to five decimals for $x=0.0(0.1)5.0$, $\alpha=0.1\pi(0.1\pi)8.0\pi$. With the help of these tables the values for negative α and x can also be obtained. Table II gives values of $\mathrm{Rr}(u)$, $\mathrm{Ri}(u)$ and their derivatives up to the third order to six decimal places for $u=0.0(0.1)5.0$, and certain other auxiliary functions are given in Table III. These can be used to compute the functions of Table I for $x>5$ and for large values of α.

R. A. Rankin (Birmingham).

Smith, R. C. T. Conduction of heat in the semi-infinite solid, with a short table of an important integral. Australian J. Physics **6**, 127–130 (1953).

Numerical values of the integral

$$\int_0^U \exp[-\alpha(1+u^2)](1+u^2)^{-1}du$$

are tabulated for various values of α and U. The temperature function $v(x, t)$ for the solid $x\geqq 0$ initially at zero temperature, whose face $x=0$ is kept at a constant temperature up to the time $t=T$ and kept insulated thereafter, is written in terms of the above integral.

R. V. Churchill.

Rutishauser, Heinz. Automatische Rechenplanfertigung bei programmgesteuerten Rechenmaschinen. Mitt. Inst. Angew. Math. Zürich no. **3**, 45 pp. (1952).

For some time various people have been interested in the problem of mechanizing, insofar as possible, the process of coding. In this paper the author analyses such a system and considers its merits. He establishes a machine procedure for handling the various portions of an arithmetic formula such as parentheses, etc., and discusses how these may be combined to produce codes. He then treats inductive processes and the complications that they entail.

H. H. Goldstine.

★Bückner, H., Weyl, F. J., Biermann, L., und Zuse, K. Probleme der Entwicklung programmgesteuerter Rechengeräte und Integrieranlagen. Herausgegeben von Hubert Cremer, Rhein.-Westf. Technische Hochschule Aachen, Mathematisches Institut, Lehrstuhl C, Aachen, 1953. xiii+75 pp.

This book contains four speeches which were given at a colloquium in Aachen in July of 1952. One of the principal purposes of the meeting was to discuss the state of computing machine developments and the relationship of this to the construction of machines in Germany. The first paper is by H. Bückner who discusses the more important features of a machine known as the Integromat which is a form of differential analyzer. It is a less expensive version of a machine at the National Physics Laboratory at Teddington, England. In the second paper F. J. Weyl reviews machine development in the U. S. A. since the war, gives comparative characteristics of machines now in use in the U. S. A., and discusses some of their uses. The third paper by L. Biermann presents an exposition of the new machine at the Max Planck Institute of Physics. He discusses the characteristics of this machine and also some of the problems being solved on it. The last paper by K. Zuse is an exposition of his work on electromagnetical machines. He is the inventor of the machine now in use in Zurich. The paper closes with a recording of the discussion which took place after the presentation of the papers.

H. H. Goldstine.

Vogel, Th. Servomécanismes, cybernétique et information. Nuovo Cimento (9) **10**, supplemento, 166–196 (1953).

Expository paper on some fundamental problems of cybernetics, written from the viewpoint of the physicist. A servomechanism is defined as that material system capable of simulating the human being in its judgment, but not in its intellectual creative activity. Topics discussed are the transmission of a message in a linear system, filtering of noise in a linear system, and servomechanisms with nonlinear elements. Numerous references are given.

S. Ikehara (Tokyo).

Samelson, Klaus, und Bauer, Friedrich L. Optimale Rechengenauigkeit bei Rechenanlagen mit gleitendem Komma. Z. Angew. Math. Physik **4**, 312–316 (1953).

Herzberger, M. Approximate methods in mathematics. Proc. Nat. Acad. Sci. U. S. A. **39**, 853–860 (1953).

Consider a scheme of approximate representation of functions in terms of a set of functions A_ν for which the product $A_\mu A_\nu$ has the representation $\sum \alpha^\lambda_{\mu\nu} A_\lambda$. The author calls attention to the fact that under certain conditions (not stated) a matrix representation of the A_ν is possible and the formal manipulations may be thereby simplified. Application is made to the use of power series expansions for solving differential equations. The truncation of the series is equivalent to the assumption that neglected powers of the variable vanish.

A. S. Householder (Oak Ridge, Tenn.).

Thomas, L. H. A comparison of stochastic and direct methods for the solution of some special problems. J. Operations Res. Soc. Amer. **1**, 181–186 (1953).

Tasny-Tschiassny, L. The location of the roots of polynomial equations by the repeated evaluation of linear forms. Quart. Appl. Math. **11**, 319–326 (1953).

Here is described a method for the location of the roots of polynomial equations. The method is particularly suitable for "punched card" and "digital electronic" computing machines. The computation consists essentially in the evaluation of certain linear forms $\sum a_q b_q$ in which the set of quantities a_q depends on the special numerical problem in hand and the set of quantities b_q is taken from tables which have been compiled for the method.

E. Frank.

Minkiewicz, Jan. Sur la résolution approchée de l'équation du cinquième degré. Avec une remarque par M. Biernacki. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. 5 (1951), 93–96 (1953). (Polish and Russian summaries)

Frame, J. S. The solution of equations by continued fractions. Amer. Math. Monthly 60, 293–305 (1953).

This is a study of a set of continued fraction approximants P_k/Q_k for the correction ϵ of a first estimate x to a required root ξ of any given equation $\phi(x)=0$, provided that $\phi(x)$ has as many continuous derivatives as may be required in a neighborhood U of the root ξ. Let $y=\phi(x)$, $y'=\phi'(x)$, $\cdots$, $\phi(\xi)=0, v=-y/y', \beta=y''/2y', \gamma=y'''/3y'', \delta=y^{(4)}/4y''', \cdots$, and

$$\text{(A)}\quad \epsilon=\xi-x=\frac{a_1}{1+}\frac{a_2}{1+}\cdots\frac{a_k}{+1+r_{k+1}}=\frac{P_k+r_{k+1}P_{k-1}}{Q_k+r_{k+1}Q_{k-1}},$$

where

$$r_k=\frac{a_k}{1+r_{k+1}}.$$

The a_k are determined from the power series expansion in v of $\epsilon=r_1=\sum c_k v^k$ in which the c_k are certain rational functions of the first $k-1$ of the quantities β, γ, δ, $\cdots$ evaluated at x. Let $a_1=v$. By division, power series are obtained for $r_2=(a_1-r_1)/r_1=\beta v+\cdots$, $r_{k+1}=(a_k/r_k)-1$, $\cdots$. Let a_k be the first term in the series expansion for r_k. If the first estimate x is not taken at a double root of any of the first k remainders, a_k/v will be a rational function of the first $k-1$ quantities $\beta, \gamma, \delta, \cdots$, independent of v. Thus $a_1=v$, $a_2=v\beta$, $a_3=v\beta-v\gamma$, $a_4=v\beta-v\gamma(\gamma-\delta)/(\beta-\gamma)$. In general

$$(k+1)\frac{a_{k+1}}{v}=2\frac{a_2}{v}+(k-2)\frac{a_{k-1}}{v}\frac{a_k}{v}-\frac{d}{dx}\ln\frac{a_k}{v},\quad k\geqq 2.$$

The a_k and r_k are also obtained as ratios of certain determinants. The kth approximant to the error $\epsilon=\xi-x$ is defined as that rational function P_k/Q_k of $a_1, \cdots, a_k$ obtained from (A) by the replacement of r_{k+1} by 0. A comparison of the approximants $P_1/Q_1, P_2/Q_2, \cdots$ is discussed in numerical examples. A proof is given of the formula due to Tauber of the derivative of the kth iteration function

$$f_k(x)=x+P_k/Q_k,$$
$$f_k'(x)=(x+P_k/Q_k)'=(k+1)(-1)^k a_2a_3\cdots a_{k+1}/Q_k^2.$$

By this formula the error in the use of $f_k(x)=x+P_k/Q_k$ as an approximation to the root ξ of $\phi(x)=0$ can be estimated. It is shown that if the numerators $a_2, \cdots, a_{k+1}$ are positive, the function $f_k(x)$ gives a better approximation to ϵ than is obtained from the kth partial sum of the corresponding power series. *E. Frank* (Chicago, Ill.).

Waadeland, Haakon. On some transcendental equations. II. Norske Vid. Selsk. Forh., Trondheim 25 (1952), 42–45 (1953).

For part I see same Forh. 24, 16–19 (1952); these Rev. 14, 150. For the equations $\cosh z \cos z=\pm 1$, all positive roots may be found from a power series obtained by Lord Rayleigh [The theory of sound, v. 1, 2nd ed., Macmillan, 1929]. The convergence of this series is considered here.

E. Frank (Chicago, Ill.).

Waadeland, Haakon. On some transcendental equations. III. Norske Vid. Selsk. Forh., Trondheim 25 (1952), 46–49 (1953).

Here is considered the convergence of the power series which give the positive roots of the equations $\operatorname{tg} x=x$ and $\operatorname{tg} x=2x$. *E. Frank* (Chicago, Ill.).

Forsythe, George E. Solving linear algebraic equations can be interesting. Bull. Amer. Math. Soc. 59, 299–329 (1953).

The advent of high speed automatic digital computing machines has revived interest in the problem of solving systems of linear equations. During the last few years old methods have been rediscovered and they, together with new methods which have been developed, have been evaluated in the light of present engineering realities. The author has performed valuable service to numerical analysis by his bibliographical work (involving about 450 titles) and his classification of the methods [Simultaneous linear equations and the determination of eigenvalues, Nat. Bur. Standards Appl. Math. Ser. no. 29, Washington, D. C., 1953, pp. 1–28].

In this invited address Forsythe begins by discussing various ways of classifying the methods. The most usual distinction made is between "direct" and "iterative methods". However, for rounded off calculations all calculations become iterative. An iterative process is called "linear" if a linear function defines the iteration for the solution vector: $x_{k+1}=F_k(x_0, x_1, \cdots, x_k)$. If F_k depends only on x_k then the process is called "of first degree". If further F_k is independent of k then the process is "stationary". The convergence of the stationary linear process depends on whether the characteristic roots of a certain matrix are all of absolute value <1. This introduces a number of problems, namely to test whether a matrix has such characteristic roots and if so, how to "accelerate" the convergence. A class of processes, not all linear, are the "general least-square" methods. They include the gradient method (or method of steepest descent), and the optimum gradient method. Possible accelerations are discussed. This leads to one of the most attractive methods found recently, independently, by several authors: the conjugate gradient method which is a finite iteration. It is non-linear and stationary. The general problem of truncation and round-off errors is discussed; the latter have mainly been studied for elimination. Errors inherent in the system of equations lead to the concept of "ill-conditioned" systems and the more general concept of "condition" of a system of linear equations. Finally, the influence of the computing equipment is discussed. Analogue machines are not discussed, but reference to relevant literature is given. For desk-machine computing Gaussian elimination seems to remain the most convenient method. Various remarks concerning methods suitable for automatic digital computers, in particular for SWAC conclude the report. A bibliography of 133 titles is added. The author has certainly shown that the solution of systems of linear equations still presents a challenge not only to the computer, but also to the pure mathematician. (Printing errors communicated by the author: p. 304, line 5 from bottom: delete "stationary"; p. 326, line 2 from top: instead of 183 read 133; 302, line 7 from bottom instead of "equalities" read "inequalities".)

O. Taussky-Todd.

Lotkin, Mark, and Remage, Russell. Scaling and error analysis for matrix inversion by partitioning. Ann. Math. Statistics **24**, 428–439 (1953).

There are well known formulas relating the inverse of a matrix of order $k+1$ to that of a principal minor of order k. The authors consider only positive definite matrices, and assume a machine representing only numbers of magnitude less than unity, describing for this situation an effective inductive procedure, with scaling, and obtaining estimates of the buildup of rounding errors. One of the advantages of the method, as pointed out by the authors, is that when the approximate inverse of a minor has been obtained by the direct formulas, one can improve it by iteration, if desirable, before proceeding to invert the minor of next higher order. Also the determinant falls out as a simple byproduct.

A. S. Householder (Oak Ridge, Tenn.).

Dwyer, Paul S., and Waugh, Frederick V. On errors in matrix inversion. J. Amer. Statist. Assoc. **48**, 289–319 (1953).

The paper mainly gives methods for numerically estimating the exact (not linearized) inherent error in the inverse A^{-1} of a matrix A, resulting from errors in A. There is a brief discussion of the computational (round-off) error, and a bibliography of 31 American and European titles. A numerical example of order 4 illustrates the several techniques.

Concerning the inherent error: If $T=A+E$ and $C=A^{-1}$, then $T^{-1}=C(I+EC)^{-1}$, whence

$$D=T^{-1}-C=-CEC(I+EC)^{-1}$$
$$=-CEC[I-EC+(EC)^2-\cdots].$$

From such formulas as the last the discrepancy D between the true inverse T^{-1} and the calculated inverse C can be bounded. The linearized part of D is $-CEC$. Norm bounds like $N(D)\leq N(C)N(EC)[1-N(EC)]^{-1}$ are often too loose. The authors show how to get the best possible bound for any preassigned element d_{rs} of D, given only that each $|e_{ij}|<\eta_{ij}$, where the η_{ij} are prescribed. The real problem is to determine the signs of the n^2 elements e_{ij} which yield the extreme values of d_{rs}. Once this is done, several methods are given to determine d_{rs}. An important lemma is that if E is so small that the signs of the elements of T^{-1} and C agree, then the extreme values of the exact discrepancies d_{ij} and the linearized discrepancies occur for the same values of the e_{ij}. Special attention is given to the case $\eta_{ij}=\eta$, and to A for which there exist u_i and v_j such that sign $c_{ij}=$ sign u_iv_j.

G. E. Forsythe (Los Angeles, Calif.).

Weissinger, Johannes. Verallgemeinerungen des Seidelschen Iterationsverfahrens. Z. Angew. Math. Mech. **33**, 155–163 (1953).

The author expresses a proof of the convergence of the Seidel process for solving the system $\mathfrak{A}\mathfrak{x}=\mathfrak{r}$, $\mathfrak{A}$ positive definite, in terms of a known fixpoint theorem for linear transformations in Banach spaces [for statement and history of which see review of Weissinger article, Math. Nachr. **8**, 193–212 (1952); these Rev. **14**, 478]. In this way the author also proves the convergence of related group-iteration methods, and methods for matrices $\mathfrak{A}$ which are semidefinite or nearly symmetric. The Seidel method is similarly extended to the solution of Fredholm integral equations of the second kind with continuous kernels. The basic lemma of the proof shows that the norm of the Seidel operator in the $\mathfrak{A}$ metric is less than one.

G. E. Forsythe.

Beljajew, N. The method of junction figures for solving normal equations in the adjustment of large nets. Bull. Géodésique 1953, 95–137 (5 plates) (1953). (Spanish. German, English, French, and Italian summaries)

Wegner, Udo. Contributi alla teoria dei procedimenti iterativi per la risoluzione numerica dei sistemi di equazioni lineari algebriche. Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. Sez. I. (8) **4**, 1–48 (1953).

This is a translation by Maria de Schwarz of parts I and II of a 3-part exposition of some of the author's extended research and study on iterative methods in linear algebra. The author has new results, and a new approach to known work. There are detailed proofs of all formulas, and many illustrative examples with matrices of orders 3 or 4. There are about 25 references.

Part I deals with bounds for the eigenvalues $\lambda_\nu{}^A$ of a real symmetric matrix A of order n, and consequences for nonsymmetric matrices. If $A>0$ (positive definite), the author proves that $\lambda^A{}_{\min}\geq d(n-1)^{n-1}s^{1-n}$ and that

$$\lambda^A{}_{\min}\geq s-[s^n-nd(n-1)^{n-1}]^{1/n},$$

where $d=|A|$ and $s=$ trace (A), and there are various refinements of these results. These are shown to yield good bounds for $\lambda^A{}_{\min}$ when applied to $A-cE$, $E=$ identity, when good lower bounds for $|A-cE|$ can be found. Let $A>0$ be partitioned into a matrix of submatrices: $A=[A_{kl}]$ $(k,l=1,\cdots,r)$, with all A_{kk} square. The author proves that $|A|\leq|A_{11}||A_{22}|\cdots|A_{rr}|$. If all $a_{ij}>0$, he discusses the convergence of the ordinary iterative process $\mathfrak{x}_r=A\mathfrak{x}_{r-1}$ to the dominant eigenvector of A. For symmetric but not necessarily definite matrices A, three types of bounds for a λ_k are given which require some knowledge of one or two neighboring λ_ν; one of these is the following result: Assume $\lambda_n\leq\cdots\leq\lambda_{k+1}<\mu<\lambda_k\leq\cdots\leq\lambda_1$. Then $\lambda_k-\mu$ is the minimum of the positive values of $\mathfrak{x}'(A-\mu E)^2\mathfrak{x}/\mathfrak{x}'(A-\mu E)\mathfrak{x}$.

In part II the author gives the linear iterative process for solving a system $A\mathfrak{x}=\mathfrak{z}$ in its most general form. Mostly he deals with iterations of type $R\mathfrak{x}_{m+1}+S\mathfrak{x}_m=\mathfrak{z}$, where $A=R+S$, $|R|\neq0$. Several possible choices of R are discussed, and there are some new sufficient conditions for convergence, e.g.: (i) $4\lambda^{S'S}_{\max}<\lambda^{A'A}_{\min}$; or (ii) the symmetry and positive definiteness of both $R+S$ and $R-S$. Known conditions of Collatz, Hertwig, Bückner, Cimmino, and others are derived from a unified point of view. The Schulz iteration $X_{m+1}=X_m(2I-AX_m)$ for getting A^{-1} is derived (A not necessary symmetric) and some X_0 guaranteeing convergence are given.

Misprint: Formulas f) and g) on page 23 should contain $\lambda_k-\mu$, not $\lambda_{k-1}-\mu$. The last sentence on page 47 suggests the false conclusion that the Schulz algorithm converges linearly, instead of quadratically.

G. E. Forsythe.

Tordion, Georges. Die Simpsonsche Formel für die zweifache Integration. Elemente der Math. **8**, 86–88 (1953).

The m-fold integral

$$F(x_n)=\int_{x_0}^{x_n}\int_{x_0}^{x}\cdots\int_{x_0}^{x}f(x)dx\cdots dx$$

is calculated by means of a suitable approximation $g(x)$ to $f(x)$ and by means of the Laplace-transformation formula $F(x)=L^{-1}(p^{-m}Lg)$. This is carried out for the special case, that the function curve of $g(x)$ consists of parabolic arcs, which coincide with $f(x)$ in equidistant pivotal points. This

leads to Simpson's formulae. The reviewer believes that the Taylor-formula

$$F(x_n) = \int_{x_0}^{x_n} \frac{(x_n - t)^{m-1}}{(m-1)!} f(t)dt$$

makes it superfluous to apply and to invert a Laplace-transformation. *H. Bückner* (Holloman, N. M.).

Tyler, G. W. Numerical integration of functions of several variables. Canadian J. Math. **5**, 393–412 (1953).

The author's principal concern are formulae for the integral $I = \int_{-a}^{a}\int_{-b}^{b} F(x, y)dxdy$ with $F(x, y) = \sum_{i+k\leq n} A_{ik}x^iy^k$. To this end, he applies the method of calculating a weighted sum $I_1 = 4ab\sum_{k=1}^{m} R_kF(x_k, y_k)$ with certain weights R_k and certain abscissae x_k, y_k. It is not assumed that $x_k \neq x_i$ for $k\neq i$. The condition $I = I_1$ for all polynomials F with constant n leads to relations between weights and abscissae. Solutions of these relations lead to quadrature formulae. The author describes such formulae for $m=4$, $n=1$; $m=5$, $n=3$; $m=13$, $n=5$; $m=12$, $n=7$. The method is also applied to parabolic regions of integration and to some multiple integrals. *H. Bückner* (Holloman, N. M.).

Faedo, Sandro. Sulla maggiorazione dell'errore nei metodi di Ritz e dei minimi quadrati. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **14**, 466–470 (1953).

In earlier work the author [Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **6**, 73–94 (1947); these Rev. **9**, 106] and Picone [Rend. Circ. Mat. Palermo **52**, 225–253 (1928)] considered the problem of finding the order of approximation to the solution of

(1) $L[y] = (B(x)y')' + A(x)y = f(x), \quad y(a) = y(b) = 0,$

when a least-squares method in terms of finite linear combinations $Y_n(x)$ from a complete set $\{\varphi_k(x)\}$ is employed for the approximation. The author shows by an extremely simple example, where $B\equiv 1$, $A\equiv 0$, $f\equiv x$, that if

$$\varphi_k(x) = \sin k\pi(x-a)/(b-a),$$

then $\rho_n = \max |Y_n(x) - y(x)| \geq Kn^{-4}$. This demonstrates that a general theorem of Picone on Ritz approximations for (1) cannot be materially improved. *C. R. DePrima.*

Collatz, L. Fehlerabschätzungen zum Iterationsverfahren bei linearen und nichtlinearen Randwertaufgaben. Z. Angew. Math. Mech. **33**, 116–127 (1953).

The paper concerns the estimation of the truncation error at a point resulting from the finite termination of an iterative solution of a linear or nonlinear boundary-value problem in n dimensions. The typical linear problem is to find u such that $L[u] = p(x)u + q(x)$ for $x \in B$, with homogeneous b.c. (boundary conditions). Nonhomogeneous b.c. are reduced to homogeneous b.c. Let $u_{n+1} = Tu_n$ if $L[u_{n+1}] = pu_n + q$ and u_{n+1} satisfies the b.c. If the class of admissible functions is so normed that $\|Tf_1 - Tf_2\| \leq K\|f_1 - f_2\|$, then it is known that (*) $\|u_n - u\| \leq K(1-K)^{-1}\|u_{n-1} - u_n\|$, where u is the solution. The author discusses the selection and use of norms which: (i) lead to K small enough to make (*) useful; and (ii) furnish pointwise bounds on $u_n(x) - u(x)$.

The usual mean-square norms fail in regard to (ii). A norm proposed for the linear problem is $\|f\| = \sup_{x\in B} |f(x)|/Z(x)$, where $Z(x)$ is an eigenfunction of $L[z] = \lambda z$, positive inside B. This leads to K of form $\max_B |p|/\lambda_1$, where $L[Z] = \lambda_1 Z$. For quasi-linear problems a norm proposed is

$$\|f\| = \sup_{x\in B} |f(x)|/W(x),$$

where $W(x)$ is a suitable function positive inside B. Nonlinear problems are also treated by (*). There are numerical examples (for $n=1$) of estimates for each class of problem treated. *G. E. Forsythe* (Los Angeles, Calif.).

✶**Collatz, Lothar. Sulla maggiorazione dell'errore nel problema di Dirichlet per le equazioni alle derivate parziali di tipo ellittico.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 68–71. Casa Editrice Perrella, Roma, 1953.

A summary of results which have since been published in greater detail by the author in Z. Angew. Math. Mech. **32**, 202–211 (1952) [these Rev. **14**, 588]. *W. Wasow.*

Gossard, Myron L. An iterative transformation procedure for numerical solution of flutter and similar characteristic-value problems. NACA Rep. no. **1073**, ii+45 pp. (1952).

"Supersedes NACA Tech. Note no. **2346** (1951)" [these Rev. **13**, 166].

Thom, A. The arithmetic of field equations. Aeronaut. Quart. **4**, 205–230 (1953).

The author describes clearly the derivation and application of some finite difference formulae (the "twelve", "twenty", and "hundred") useful for solving the equation $\nabla^2\psi = f(x, y)$ with appropriate boundary conditions. He also discusses the equation $\nabla^4\psi = f(x, y)$ which he prefers to treat as a system $\nabla^2\zeta = f(x, y)$, $\nabla^2\psi = \zeta$, where he indicates an appropriate technique for obtaining the boundary values of ζ. The presentation is complete with well chosen and carefully worked out examples. The author describes special devices (not relaxation) which can be employed to shorten the labor. He explains how to treat problems with singularities (i.e., corners in a flow, etc.) and how to select the most effective finite difference formula in such cases. The method of computing the conjugate function is derived and its application described for some of the standard sets of conjugate functions that may be used in two-dimensional flow problems. The determinations of the slow two-dimensional flow of a viscous fluid between parallel planes having small symmetric bumps at opposite points is worked out in great detail as an illustration for the equation $\nabla^4\psi = f(x, y)$. (The perturbation about the non-viscous flow is computed.)

Practical rules for the estimation of rounding off errors are given for various finite difference formulae. The author notes in concluding, "the formulae and methods given in the paper are intended for desk work. With electronic calculating machinery designed for this type of work it is almost certain that the original Liebmann-method would be the most suitable. Nevertheless the use of the double field for $\nabla^4\psi$ equations would still be an advantage." *E. Isaacson* (New York, N. Y.).

Laurikainen, Kalervo V., und Euranto, Erkki K. Beiträge zur numerischen Behandlung der Schrödinger-Gleichung im Falle des Yukawa-Potentials. Z. Angew. Math. Physik **4**, 155–158 (1953).

Approximate eigenvalues and eigenfunctions for the radial Schroedinger equation in the case of the Yukawa potential are derived from the variational principle by using various types of trial wave functions satisfying the appropriate boundary conditions. Earlier calculations for the s-state [L. Hulthén and K. V. Laurikainen, Rev. Modern Physics **23**, 1–9 (1951); these Rev. **12**, 862] have here been general-

ized to p- and d-states, and tables for the lowest eigenvalues, as well as some asymptotic expansions, are given.

P. O. Löwdin (Cambridge, Mass.).

Good, R. H., Jr. The generalization of the WKB method to radial wave equations. Physical Rev. (2) **90**, 131–137 (1953).

In considering the Dirac and the Schroedinger radial wave equations for a particle in a spherically symmetric potential field and the ordinary one-dimensional wave equation, the author shows that a generalized WKB-treatment can be carried out for the three cases in a uniform way by using half-integer Bessel-functions instead of the conventional exponentials. In the radial problems no connection formulas are needed across the turning points, but the approximations give also especially poor results near these points. Since the approximate wave functions become identical with the exact solution for a free particle, the accuracy of the approximations generally increases with increasing energy. *P. O. Löwdin* (Cambridge, Mass.).

Suyama, Yukio, and Nakamori, Kanzi. On numerical solution of the integral equation of Volterra type. Mem. Fac. Sci. Kyūsyū Univ. A. **6**, 121–129 (1952). (Esperanto)

Application of the Runge-Kutta method to the integral equation $u(x)=f(x)+\int_{x_0}^{x}K(x, t, u(t))dt$ with $f(x)$ and $K(x, t, u)$ having continuous derivatives up to a sufficiently high order. A numerical formula is derived by differentiation of the given equation and by means of Taylor formulae.

H. Bückner (Holloman, N. Mex.).

Rodríguez-Salinas, Baltasar. On various procedures for the determination of the periods of tides and their prediction at a particular place. Revista Acad. Ci. Madrid **46** (1952), 441–457 (1953). (Spanish. English summary)

The author derives limiting equations for fitting a given function by a trigonometric sum with adjustable coefficients and frequencies. Two problems are considered: minimization of the sum of squares of residuals of the given values, and minimization of a weighted integral of the Laplace transform of these residuals. In practice these methods encounter difficulties due to near indeterminacy associated with the multiplicity of the admitted frequencies, and to noise and autocorrelations present in many of the data that arise in natural phenomena.

A. Blake (Buffalo, N. Y.).

Zwinggi, Ernst. Beiträge zum Zinsfussproblem. Bl. Deutsch. Ges. Versicherungsmath. **1**, no. 3, 105–113 (1952).

Let $P(\delta)+\delta$ denote the reciprocal of a probability generating function (argument δ). The author has previously expressed the ratio $\ln\{P(\delta)/P(\delta_0)\}$ as a Taylor series in $\delta-\delta_0$ [Skand. Aktuarietidskr. **33**, 88–97 (1950); these Rev. **12**, 135]. He now writes this ratio as a series in $e^{-\delta}-e^{-\delta_0}$. In the particular case considered, the numerical approximations provided by the first two terms show an improvement over the corresponding values of the earlier paper.

H. L. Seal (New York, N. Y.).

Richard, P.-J. La représentation analytique des tables de mortalité. Exposé historique et didactique de la méthode générale d'Albert Quiquet. Bull. Trimest. Inst. Actuaires Français **51**, 177–268 (1952).

Jecklin, Heinrich. Über gewisse Approximationen der Versicherungsmathematik. Bl. Deutsch. Ges. Versicherungsmath. **1**, no. 2, 3–16 (1951).

MECHANICS

Zinov'ev, V. A. Kinematic analysis of spatial mechanisms. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **10**, no. 42, 52–99 (1951). (Russian)

The kinematical theory of machines and mechanisms is usually taught graphically. For precision, however, it is necessary to resort to analytical or numerical methods. This is difficult enough in the plane, but this work attempts to illustrate some of the possibilities in the analytical treatment of spatial mechanisms. The foundations of vector methods in three dimensions are derived. As an example, the relative positions of the links of a closed hinged chain of six links are shown to satisfy a set of twelve equations of which two are linear and the others are quadratic. A numerical example is solved by successive approximation by making use of the given lengths, twists and offsets of the links. Other examples involving, in addition to hinges, ball-and socket-joints and screw pairs are given.

M. Goldberg (Washington, D. C.).

Sbrana, Francesco. Sul teorema di unicità per le equazioni differenziali della meccanica. Boll. Un. Mat. Ital. (3) **8**, 123–127 (1953).

Assuming that the constraints and forces of a mechanical system satisfy certain simple continuity conditions, the author proves that if the initial values of the velocities and accelerations are zero, the system remains at rest. Exceptional cases in which the differential equations of motion and the initial values of the coordinates and velocities do not suffice to determine the motion uniquely are considered, and the question is raised as to which of the mathematically possible motions actually ensues. It is suggested that this is the motion for which the work done by the active forces is least. *L. A. MacColl* (New York, N. Y.).

Tzénoff, Iv. Détermination des forces intérieures dans un corps solide en équilibre dont les déformations sont négligeables. Annuaire [Godišnik] Fac. Sci. Phys. Math., Univ. Sofia, Livre 1, Partie I. **47**, 75–91 (1951). (Bulgarian. French summary)

The general remarks in this paper pertain to elementary matters concerning the equations for the forces exerted by one part of a rigid body on another. Among the examples there is one treating a circular, homogeneous helix under the action of its own weight and certain forces at its terminal cross section. The equations for the forces in the internal cross section are written in terms of their derivatives with respect to the length of arc of the helix, and completely integrated for certain external loads. *A. W. Wundheiler.*

Krutkov, Yu. A. On a new type of quasi-coordinate. Doklady Akad. Nauk SSSR (N.S.) **89**, 793–795 (1953). (Russian)

To understand this note [a revision of Izvestiya Akad. Nauk SSSR. Otd. Fiz.-Mat. Nauk (7) **1928**, 549–572], it is necessary to realize that "relative coordinates" of a point P are coordinates only if the position of the moving reference frame F is a function of time and the position of P, at the most. The author concentrates on the case when the angular velocity of F is a function of time, position, and relative velocity of P, as it may happen when a vehicle F pursues a moving point P. For this case he writes the equations of motion in a Lagrange-type form.

A. W. Wundheiler (Chicago, Ill.).

Manacorda, Tristano. Il moto di un corpo di massa variabile. Rivista Mat. Univ. Parma **3**, 361–373 (1952).

The study of the dynamics of rockets has directed serious attention to the motion of a body with variable mass. J. B. Rosser, R. R. Newton, and G. L. Gross [Mathematical theory of rocket flight, McGraw-Hill, New York, 1947; these Rev. **9**, 108] have based their treatment of rocket motion on two relations which are substantially the cardinal equations of motion for a body with variable mass. In the paper under review these two cardinal equations are derived in a different but rigorous way. The new derivation of the second equation is considerably shorter than that given by Rosser, Newton, and Gross. These equations are applied to a simple case of rocket motion in a vacuum where the rocket's axis of revolution coincides with the z-axis of the moving trihedral for the duration of the motion.

E. Leimanis (Vancouver, B. C.).

Madejski, J. The theory of the oscillational governors. Arch. Méc. Appl., Gdańsk **3**, 371–417 (1951). (Polish. English summary)

The paper discusses the performance of (marine-engine) inertial relay governors of the "on-off" type. A shaft-driven rocker carries a weight ("relay") which can revolve, relative to the rocker, about an axis either parallel (class I) or perpendicular (class II, exemplified by the Aspinall type) to the axis of oscillation of the rocker. Some of the discussion is no more than program.

Let α be the angular displacement of the relay relative to the rocker, ϕ that of the rocker, a known function of the shaft angle ψ, and let $\dot\psi=\omega$. The force exerted by the relay on the cut-off lever (at contact) is proportional to $\ddot\alpha$, hence the maximum ω for action will occur at $\dot\omega$, while the minimum ω will occur at the maximum ϵ of $\dot\omega$ (assumed given). For both classes of regulators and $\dot\omega=0$, the author obtains (after considerable linearization) the equation $\alpha''+L\alpha=M\psi''+A_0$, where the primes indicate differentiation with respect to the shaft angle ψ, and L, M, A_0 are constants. Some numerical examples are given to justify the usefulness of the approximation.

Another equation, linear in α'', α', α, but with coefficients depending on ψ, is derived for the case of constant $\dot\omega=\epsilon$. For negligible rocker velocities, this equation can be integrated by quadratures involving Bessel functions but this solution is useless for design purposes. This makes the author turn to rather obscure approximations leading to an estimate of the ratio of minimum and maximum ω for action. Next the permissible maximum initial ($\alpha=0$) tension of the spring pressing the cut-off lever against the relay, is connected with ω, and the angular shaft displacement from $\alpha=0$ to the instant of action. The remainder of the paper (too difficult to follow for this reviewer) is devoted to the study of "the process of regulation, the degree of nonuniformity of engine operation", the influence of trim, list, rolling, and pitching. There is a remark about the emergency device that prevents racing on sudden load reduction.

A. W. Wundheiler (Chicago, Ill.).

★Pöschl, Theodor. Sull'estensione del concetto di oscillazioni principali ai sistemi non-lineari a più gradi di libertà. Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 542–549. Casa Editrice Perrella, Roma, 1953.

By a principal oscillation of a dynamical system is meant an oscillation in which all particles pass through their equilibrium positions simultaneously, and attain their maximum departures from equilibrium simultaneously. In this note the author discusses an elementary example of such oscillations in detail, and suggests a procedure for studying more general cases. *L. A. MacColl* (New York, N. Y.).

Minakov, A. P. Equilibrium of an ideally flexible cord on a rough surface. Foundations of the theory of winding and unwinding of a cord. Moskov. Gos. Univ. Učenye Zapiski **154**, Mehanika **4**, 241–266 (1951). (Russian)

Since the paper does not consider the centrifugal force (relative to the spinning spool), it applies to winding only if this force is negligible. For a fixed point on a fixed surface, the three standard equations of equilibrium contain six variables: the angle θ between the principal normal to the cord curve and the surface normal; the friction-force magnitude F per unit length; the angle α between the friction force and the tangent to the curve; the normal force N per unit length; the tension T of the cord; and the derivative of T with respect to the arc length. The problem is indeterminate but the author states, repeatedly, that these equations solve it. If κ is the angle between the full reaction and the surface normal, the expression $T=T_0\exp[\int\tan\kappa(ds/\rho)]$ is easily obtained. This generalized Euler formula yields easily a necessary equilibrium condition (1) $\tan\theta\leq\mu$ where μ is the coefficient of friction; the author uses it as a sufficient condition. If the equality holds, there is limiting equilibrium, $\alpha=\pi/2$, the tension is constant, $N=T/R$ (R is the radius of curvature of the tangent normal section), and the problem becomes determinate. If two points of a cord are given, a region of equilibrium is defined.

The bulk of the paper is devoted to fairly simple geometry producing relations for curves making a constant "Clairaut angle" β with the meridians of a surface of revolution, and curves making a constant angle θ with the normal to such a surface. These formulas are further specialized for circular cylinders and cones. Four special cases are thus defined, and for which the expression for T is derived. The condition (1) is offered as a criterion that no slipping will occur. It restricts the direction of the thread relative to the spool, and the ratio of the spinning speed ω to the velocity v of the last contact point.

The paper contains confusing mentions of "Amonton's friction law" which sound as if the author believed that the ratio F/N was determined even in the non-limiting case. It discusses "constant winding velocity" although only the ratio v/ω is relevant to equilibrium. Finally, deep concern is shown for the "correct" definition of the "angle of girth" between the cord and the spool which does not significantly enter into any tension formulas. Without justification, the author submits, as the useful concept, the length of arc of

the spherical indicatrix for the portion of the cord in contact with the spool, as traditionally used in defining the curvature of twisted curves. *A. W. Wundheiler.*

Hydrodynamics, Aerodynamics, Acoustics

Cummins, William E. The forces and moments acting on a body moving in an arbitrary potential stream. The David W. Taylor Model Basin, Washington, D. C., Rep. **780**, v+47 pp. (1953).

It is shown how the force and moment acting on a body with an arbitrary motion through an inviscid incompressible fluid subject to a time-dependent potential flow can be found if the body can be represented by a system of singularities placed within the body. The force can be split into three components. The first, which would be the total force if the instantaneous flow were steady, is found in terms of general singularities. The second, which depends upon the change with time of the singularity system generating the surface of the body, is found to be a function of the sources and doublets in the singularity system but not of the higher order singularities. The third is the force which would be required to generate the given motion of the body in a vacuum if the body were to have the same density as the fluid. An investigation of the moment is also given, but except in the case of the first component the analogy with the force components does not occur.

L. M. Milne-Thomson (Greenwich).

Shibaoka, Yoshio. An example of the flow including a free stream line in the perfect fluid. J. Inst. Polytech. Osaka City Univ. Ser. B. Physics **3**, 53–57 (1952).

Le schéma suivant d'un écoulement plan et à potentiel d'un fluide incompressible est considéré: La frontière du domaine fluide est constituée par deux demi-droites parallèles, reliées par une ligne à vitesse constante, et le mouvement est dû à la présence d'une source à l'infini et d'un puits sur l'une des parois. L'étude est faite par la méthode de l'hodographe. Il est montré qu'il existe une solution du problème posé, et les équations de la ligne libre sont explicitées. Il n'y a pas de solution lorsque la seule singularité (doublet) est à l'infini. *R. Gerber* (Toulon).

Sretenskiĭ, L. N. On waves on the surface of separation of two flows of a liquid flowing at an angle to each other. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1952**, 1782–1787 (1952). (Russian)

Let a fluid of density ρ fill the region $z<0$ and a fluid of density ρ' the region $z>0$. The lower fluid moves with velocity c in the direction OX and the upper fluid with velocity c' in a direction making an angle ϑ with OX. The author investigates gravity waves at the interface $z=0$. For motion with a velocity potential, linearized boundary conditions are derived for the interface. Velocity potentials of the form

$$\Phi=A\exp[kz+i(mx+ny)],\quad \Phi'=A'\exp[-kz+i(mx+ny)]$$

are assumed and the resulting interface studied in some detail for its dependence on m and n. *J. V. Wehausen.*

Sretenskiĭ, L. N. Spatial problem of determination of steady waves of finite amplitude. Doklady Akad. Nauk SSSR (N.S.) **89**, 25–28 (1953). (Russian)

The author considers three-dimensional progressive waves of finite amplitude in infinitely deep water. He assumes a velocity potential of the form

$$\begin{aligned}\varphi(x,y,z)=&-vx\\&+\{\alpha_{11}\exp k_{11}z\cos ny+\alpha_{13}\exp k_{13}z\cos 3ny+\cdots\}\sin mx\\&+\{\alpha_{20}\exp 2k_{20}z+\alpha_{22}\exp 2k_{n}z\cos 2ny+\cdots\}\sin 2mx+\cdots,\end{aligned}$$

expands the α_{ij} and coordinates (Lagrangian) in powers of a small parameter ϵ, substitutes in Bernoulli's equation, compares coefficients, etc. He finally obtains equations which can be used to relate the wave velocity, amplitude, and lengths in the two directions. There is mass transport as for plane waves. [Cf. Fuchs, Gravity waves, pp. 187–200, Nat. Bur. Standards Circular 521, Washington, D. C., 1952; these Rev. **14**, 1028.] *J. V. Wehausen.*

Gotusso, Guido. Una proprietà delle onde sulla superficie di un liquido. Boll. Un. Mat. Ital. (3) **8**, 36–40 (1953).

Using the result of an earlier paper [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **10**, 130–132 (1951); these Rev. **13**, 395], the author shows that, of all possible gravity-wave motions of an incompressible inviscid infinitely deep fluid with given value of the free surface on the boundary, that one occurs for which

$$\int_\sigma\left[\left(\frac{\partial v_x}{\partial t}\right)^2+\left(\frac{\partial v_y}{\partial t}\right)^2-\left(\frac{\partial v_z}{\partial t}\right)^2\right]_{z=0}d\sigma$$

is stationary. Here v_x, v_y, v_z are the components of the velocity vector. The linearized free-surface boundary conditions are assumed. *J. V. Wehausen* (Providence, R. I.).

Kostyukov, A. A. On the wave resistance of a caravan of ships. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 33–38 (1953). (Russian)

Michell's integral for the wave resistance of a "thin" ship is given by

$$R_w=\frac{4\rho g^2}{\pi v^2}\int_1^\infty(J_1^2+J_2^2)\frac{\lambda^2d\lambda}{\sqrt{\lambda^2-1}},$$

$$J_{1,2}=\int_0^T\exp(-\nu\lambda^2\zeta)d\zeta\int_{-L/2}^{L/2}{\cos \atop \sin}(\nu\lambda\xi)\frac{\partial f}{\partial\xi}d\xi,$$

where $y=\pm f(\xi,\zeta)$ is the equation of the hull, L and T are length and draft, v is velocity, g is acceleration of gravity, ρ is density, and $\nu=g/v^2$. A caravan of ships, one behind the other, all moving with the same velocity, may be treated as one long ship with zero width in the regions between ships. The author investigates the form of Michell's integral, R_{wn}, for n congruent equally spaced ships, with l the distance between centers. He finds

$$R_{wn}(l)=n\int_\nu^\infty G(u)du+2n\sum_{k=1}^{n-1}\int_\nu^\infty\left(1-\frac{k}{n}\right)\cos kluG(u)du$$

where

$$G(u)=4\rho v^2\pi^{-1}[J_1^2(u^2/\nu)+J_2^2(u^2/\nu)]u^2(u-\nu^2)^{-1/2}.$$

From this he finds $R_{wn}<n^2R_{w1}$, $\lim_{l\to\infty}R_{wn}=nR_{w1}$, as one might expect, and an asymptotic formula for large n, $R_{wn}\sim 2n\pi l^{-1}\sum_{k=-\infty}^{\infty}G(2k\pi/l)$. *J. V. Wehausen.*

Liu, Hsien-Chih. Beitrag zur Kenntnis der Eigenschwingung einer idealen Flüssigkeit in kommunizierenden Röhren. Z. Angew. Math. Physik **4**, 185–196 (1953).

The tube in question is bent and so oriented that a liquid may be held in it. The liquid column oscillates under the influence of gravity. When the parts of the tube through which the two liquid surfaces move are true cylinders, the motion of the liquid can be described in terms of a nonlinear

but readily integrable differential equation. This is integrated in terms of elliptic functions of the second kind. The two (in general unequal) semi-periods are studied. A phase plane representation of the trajectories is given and the oscillations are also plotted against time for several values of the parameters. *E. Pinney* (Berkeley, Calif.).

Jaeckel, K. Zur Theorie der Profile geringer Dicke und Wölbung. Z. Angew. Math. Mech. **33**, 213–215 (1953).

This is a new presentation of the first-order theory of airfoil profiles having small thickness, camber, and incidence. To obtain formulas for practical calculation in series form, elliptic coordinates are used. Reference is made to earlier work of Föttinger [Jbuch. Schiffbautechn. Gesellschaft **25**, 295–344 (1924)], Riegels [Ing.-Arch. **16**, 373–376 (1948); **17**, 94–106 (1949); these Rev. **10**, 490; **11**, 274] and others. *W. R. Sears* (Ithaca, N. Y.).

***Polubarinova-Kočina, P. Ya. Teoriya dviženiya gruntovyh vod. [Theory of motion of ground water.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1952. 676 pp. 13.85 rubles.

The first two chapters of this book describe the physical and mathematical fundamentals of ground-water flow. Discussions of porosity, capillarity, states of ground water, D'Arcy's law and limits of its applicability, and nonlinear filtration laws are included. The complex potential is introduced, boundary and interface conditions are discussed and illustrated.

In the next eight chapters various known methods are used to find flow from canals and into drainage systems of varied shapes and configurations, under dams and other barriers, often with unusual conditions of evaporation, back-flow, inclined layer of underlying hard-pan, nonhomogeneous ground and anisotropic ground. Interspersed where required are found discussions of the mathematics used, such as conformal mapping, semi-inverse and inverse methods, theory of functions and linear differential equations. Topics from fluid mechanics, such as the hodograph and the use of sources and sinks, are also included.

Chapter 11 describes graphical, numerical and experimental methods. The graphical scheme, presented only briefly, is essentially that of Poritsky. The use of finite differences is discussed at some length, and several examples are given, among them the only axially symmetric problems in the book. Only the finite difference approximations are given, however, and nothing is said of how one proceeds to solve the equations obtained efficiently. Nevertheless, results for several elaborate problems are reproduced from Southwell [Relaxation methods in theoretical physics, Oxford, 1946; these Rev. **8**, 355]. The "experimental" methods are actually methods of simulation. Two of them, the electric field analogy and the flow tank, are straightforward models. The third is more clever, as it sets up an analog involving laminar flow of a viscous liquid between two narrowly separated parallel plates, using mathematics taken from lubrication theory. Suggestions for handling non-homogeneous and anisotropic soil conditions and non-stationary flow by this third method are given.

Chapters 12–16 are devoted to non-stationary flow of ground water, with such topics as the filtration under dams with water levels varying in a prescribed manner, influence of waves on filtration under dams, non-stationary flow in the presence of a free surface, non-stationary loss from canals, and the behavior of the water table after rain.

This book will best serve those who need an encyclopedia of solved problems, with much of the detail and pertinent mathematics provided. For the most part, Chapters 3–10 (pp. 84–486) constitute a problem book on stationary ground water flow. In Chapters 12–16, such detailed applications are rarer, general discussion is more prevalent. Notable features of the book are its numerous sketches and its extensive chapter by chapter bibliography of Russian work in this field. *R. E. Gaskell* (Seattle, Wash.).

Balčorov, H. Ya. Plane parallel flow of an ideal incompressible liquid about a porous circular cylinder with linear and quadratic law of filtration. Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk **1952**, no. 8, 73–87 (1952). (Russian)

A cylindrical shell of uniform thickness and porosity, radius a, is placed with its axis perpendicular to the flow of an ideal liquid whose velocity at infinity is constant. The author seeks the steady-state flow of liquid outside the cylinder. The solution is first written in terms of the unknown $(\partial\phi/\partial\theta)_{r=a}$, where ϕ is the velocity potential. From the relation $\int_{-\pi}^{\pi}(\partial\phi/\partial r)_{r=a}d\theta=0$, and a previous result of the same author [same Vestnik **1951**, no. 10 (unavailable)] relating $(\partial\phi/\partial r)_{r=a}$ and $(\partial\phi/\partial\theta)_{r=a}$, he derives an integral equation of the form

$$Z(\theta)=\int_{-\pi}^{\pi}f(\xi, Z)\operatorname{ctg}[(\xi-\theta)/2]d\xi$$

for $Z(\theta)=(\partial\phi/\partial\theta)_{r=a}+2aV_\infty\sin\theta$. The work is carried through for linear and quadratic laws of filtration; numerical examples are worked out for these cases, with approximate solutions of the nonlinear integral equation obtained by iteration. *R. E. Gaskell* (Seattle, Wash.).

Belyakova, V. K. The plane problem of the variation of the form of the free surface of ground water taking account of infiltration. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 373–376 (1953). (Russian)

Ground water is assumed initially to occupy the space below the free surface $y=\delta(x, 0)=f(x)$, and water is added at this surface at the rate $\epsilon(x, t)$ per unit area. The velocity potential, $\phi(x, y, t)$, is found for this case, and in addition when there is an impermeable layer at $y=-H$, with $f(x)$ and $\epsilon(x, t)$ assumed periodic in x. The problem is solved by splitting into two problems, each with only one nonhomogeneous condition, solving each and then combining results. Answers for $\phi(x, y, t)$ and $\delta(x, t)$ are given in Fourier integral form. The author fails to include the differential equation in her statement of the first boundary-value problem, and states it incorrectly in the second. *R. E. Gaskell.*

Sokolov, Yu. D. Filtration without backwater from an unlined canal of trapezoidal section in homogeneous ground. Ukrain. Mat. Žurnal **4**, 65–96 (1952). (Russian)

A canal of trapezoidal cross-section is cut in a layer of soil of low permeability (such as clay) which overlies a layer of greater permeability (sand or gravel). Characteristics of the flow are found by the method of conformal mapping, the development being complicated but straightforward. Work is carried out in considerable detail. Water loss from the canal is determined by approximate numerical methods for several cross-sections when the depth of the layer of clay is infinite, and also for chosen finite depths of the clay layer. *R. E. Gaskell* (Seattle, Wash.).

Hovanskiĭ, A. N. On the derivation of the fundamental equations of filtration of an elastic liquid in an elastic porous medium. Doklady Akad. Nauk SSSR (N.S.) 89, 241–244 (1953). (Russian)

D'Arcy's law leads to the equation

$$\nabla\cdot(\gamma\nabla P)=\mu k^{-1}\partial(m\gamma)/\partial t,$$

where P refers to the pressure of the compressible liquid of variable density γ, absolute viscosity μ, diffusing through a medium of porosity m, and permeability k. The author proceeds from this equation, writing appropriate relationships between pressure, density, and porosity so that he can reduce the equation to one containing a single dependent variable,

$$(\nabla P)^2+K_L\nabla^2P=\mu k^{-1}K_LR\partial P/\partial t,$$

where

$$R=\{K_L^{-1}-(1-m_0)(K_L^{-1}-K_C^{-1})\exp[-(P-P_0)/K_C]\},$$

and K_L and K_C are bulk moduli of liquid and solid media, and m_0 is the porosity at pressure P_0. *R. E. Gaskell.*

Ray, M. Velocity and temperature distributions in a liquid flowing over an infinite plate: boundary layer theory. Bull. Calcutta Math. Soc. 44 (1952), 137–141 (1953).

The author considers an unsteady flow of an incompressible viscous fluid over a fixed infinite plate. By introducing a similarity variable, a special solution for the velocity and temperature is found for Prandtl number both equal to, and greater than, one. As the flow at infinity is uniform, the solution can only be interpreted to mean that the viscosity begins suddenly to operate at a given instant.

Y. H. Kuo (Pasadena, Calif.).

Moore, Franklin K. Three-dimensional laminar boundary-layer flow. J. Aeronaut. Sci. 20, 525–534 (1953).

As an example of the class of problems indicated by the title, the author treats the laminar boundary layer on a circular cone at an angle of attack α relative to a supersonic stream. The details of the potential-flow field are available in tables. The boundary layer on such a body exhibits a certain similarity; i.e., flow quantities are constant along parabolas originating at the cone apex, in any meridian plane $\varphi=\text{const}$. The Prandtl number is put equal to one, a linear temperature-viscosity relation is assumed, and the Howarth and Mangler transformations are employed.

For small α the solution is found as a perturbation of the results for symmetrical flow. For large α a solution in the plane of symmetry $\varphi=0$, π alone is sought. For the bottom surface ($\varphi=0$) (at positive incidence $\alpha>0$) velocity profiles are found at various α at two Mach numbers. At the top ($\varphi=\pi$) a more complicated situation is encountered. Beyond a rather small value of α, solutions exist but are not unique. This is believed to occur because the flow converges toward the upper meridian plane at larger α and therefore the situation there is influenced by conditions at all φ; i.e., a plane-of-symmetry study is insufficient to define it. At still larger α the solution fails to exist at all; i.e., there is no solution that has acceptable asymptotic behavior. This is believed to represent separation of the circumferential flow; the flow is not of the boundary-layer type. The paper closes with a discussion of the meaning of "separation" in three-dimensional cases. *W. R. Sears* (Ithaca, N. Y.).

Cheng, Sin-I. On the stability of laminar boundary layer flow. Quart. Appl. Math. 11, 346–350 (1953).

The author examines the approximations in the linearized equations of the stability of the laminar boundary layer in a compressible fluid and shows that the use of local velocity and temperature distributions is no longer justified if accuracy beyond that in existing calculations is desired; the cross-component of the mean flow being the new element that comes up in the next approximation. *C. C. Lin.*

Tetervin, Neal. A study of the stability of the incompressible laminar boundary layer on infinite wedges. NACA Tech. Note no. 2976, 41 pp. (1953).

By using the reviewer's estimation formula, the author calculates the critical Reynolds number for the laminar boundary-layer of an incompressible flow over infinite wedges. This is done to test the following conclusion of Schlichting. In a region of falling pressure (favorable pressure gradient), a thick velocity profile can be more stable than a thin layer, although the free stream conditions are identical up to the pressure gradient. Profiles calculated by Hartree are used for this purpose. The conclusion of Schlichting is confirmed. *C. C. Lin* (Cambridge, Mass.).

Wuest, W. Näherungsweise Berechung und Stabilitätsverhalten von laminaren Grenzschichten mit Absaugung durch Einzelschlitze. Ing.-Arch. 21, 90–103 (1953).

The author gives several calculations of the neutral curve for the stability of a laminar boundary layer with suction, and compares them with earlier calculations. In the case of asymptotic suction, his value for the minimum critical Reynolds number agrees with that obtained by Pretsch using the same method, in which certain approximations are introduced in handling the inviscid solution. Their value is higher than that given by Freeman who used an exact solution of the inviscid equation in terms of the hypergeometric function. Freeman's value agrees closely with that obtained from the reviewer's estimation formula, which give values of the critical Reynolds number, for other amounts of suction, differing from the author's values by the same order of magnitude as in the case of asymptotic suction. *C. C. Lin* (Cambridge, Mass.).

***Frenkiel, F. N. Turbulent diffusion: mean concentration distribution in a flow field of homogeneous turbulence.** Advances in Applied Mechanics, vol. 3, pp. 61–107. Academic Press Inc., New York, N. Y., 1953. $9.00.

Dans une introduction, l'auteur, après quelques remarques sur la turbulence et l'agitation moléculaire, définit l'élément fluide, dont le centre de gravité est un point aléatoire P de coordonnées x, y, z et de vitesse u, v, w. Il rappelle ensuite sommairement les principes de la théorie statistique de la turbulence, et donne la définition des coefficients de corrélations au sens de Lagrange: fonctions de corrélation du point P dans deux positions successives correspondent aux instants t et $t+h$. Si la turbulence est homogène, isotrope et stationnaire, ces coefficients sont définis par une unique fonction $R(h)$ et l'on a

$$\overline{y^2}=2\overline{v^2}\int_0^t(t-\alpha)R_h(\alpha)d\alpha.$$

Discussion: cas des petites et des grandes valeurs de t; calcul de $\overline{y^2}$ pour diverses expressions analytiques plausibles de R_h.

La suite du mémoire est consacrée à l'étude de la distribution statistique du fluide diffusé, moyennant l'hypothèse que les éléments provenant d'éléments concentrés autour d'un point sont distribués suivant la loi de Gauss. Cas d'un fluide en mouvement, d'une émission continue d'éléments, d'une source rectiligne. L'équation de diffusion classique de Fick [Ann. Physik Chemie (4) 4, 59–86 (1855)]

$\partial s_0/\partial t = k\nabla^2 s_0$, k constant, ne s'applique que pour les grandes valeurs de t. Dans le cas général, k est fonction de t.

Un appendice donne quelques indications sur la diffusion en turbulence non isotrope, et même non homogène.

J. Bass (Chaville).

Loewner, Charles. Conservation laws in compressible fluid flow and associated mappings. J. Rational Mech. Anal. 2, 537–561 (1953).

Un mouvement irrotationnel d'un fluide compressible dans le plan (x, y) est gouverné par les équations

$$(1) \qquad u_y - v_x = 0, \quad (\rho u)_x + (\rho v)_y = 0.$$

Si la densité $\rho(u, v)$ est supposée être exprimée en fonction des composantes u et v de la vitesse, à l'aide de l'équation d'état et de la relation de Bernoulli, le système (1) est alors de la forme

$$(2) \qquad a_1{}^i u_x + a_2{}^i v_x + b_1{}^i u_y + b_2{}^i v_y = 0 \quad (i = 1, 2),$$

ces coefficients $a_k{}^i$, $b_k{}^i$ étant des fonctions de u et de v. Les équations (1) ont la forme particulière

$$(3) \qquad [\xi(u, v)]_x + [\eta(u, v)]_y = 0$$

où $\xi(u, v)$ et $\eta(u, v)$ sont des fonctions de u et de v ayant des dérivées continues. L'auteur appelle loi de conservation du système (2), tout couple de deux fonctions $\xi(u, v)$, $\eta(u, v)$, telles que l'équation (3) soit une conséquence de ce système (2), et le sujet du présent travail est l'étude des lois de conservation qui sont attachées à un système quelconque du type (2).

Dans une première partie, il est montré que toutes les lois de conservation, qui sont conséquence du système (2), sont données au moyen d'un système de la forme

$$(4) \qquad c_1{}^i\xi_u + c_2{}^i\xi_v + d_1{}^i\eta_u + d_2{}^i\eta_v \quad (i = 1, 2)$$

où les $c_k{}^i$, $d_k{}^i$ sont des fonctions de u et de v. De plus, la relation entre les systèmes (2) et (4) s'obtient par un principe de dualité. Si l'on considère deux lois de conservation (ξ_1, η_1), (ξ_2, η_2), on définit une correspondance entre les plans des variables x, y et χ_1, χ_2 à l'aide des formules

$$(5) \qquad \chi_1 = \int -\eta_1 dx + \xi_1 dy, \quad \chi_2 = \int -\eta_2 dx + \xi_2 dy.$$

(Dans le cas particulier des lois de conservation (1), χ_1 et χ_2 coïncident avec le potentiel φ et le débit ψ.)

La deuxième partie du travail est consacrée à l'ètude de quelques propriétés de la correspondance (5), ce qui permet de mettre en évidence, par un choix convenable de χ_1 et de χ_2, des propriétés nouvelles des écoulements d'un fluide.

Dans une troisième partie, en reprenant des idées de Lavrentieff, l'auteur introduit la notion de système (2) et de correspondance (5) "elliptiques au sens fort". En utilisant cette notion pour des correspondances (5) attachées à un écoulement subsonique, on obtient plusieurs inégalités concernant le mouvement sur la frontière; c'est là l'objet de la dernière partie.

R. Gerber (Toulon).

Basch, A. Zur Geometrie der ebenen Strömung von Gasen. Österreich. Ing.-Arch. 7, 139–143 (1953).

For steady plane adiabatic flow of an ideal compressible fluid it is known that (1) $\partial q/\partial n = kq$, and (2) $\partial q/\partial s = K/(1-m^2)$, where q = velocity, m = Mach number, s and n are distances tangent and normal to the streamlines, and k and K are the curvatures of the streamlines and the orthogonal trajectories respectively. The author, however, seems unaware of these results, for this note is devoted mainly to a proof of (1) and (2) for isentropic irrotational flows with $m \ll 1$ and the special equation of state $p = A\rho^\gamma$. From (1) and (2) he derives a construction for the vector grad log q.

J. B. Serrin (Cambridge, Mass.).

***Zaldastani, Othar. The one-dimensional isentropic fluid flow.** Advances in Applied Mechanics, vol. 3, pp. 21–59. Academic Press Inc., New York, N. Y., 1953. $9.00.

Cet article constitue une synthèse des principaux résultats relatifs à l'étude analytique des écoulements rectilignes des gaz dans le cas où ces écoulements restent partout continus. L'étude est souvent restreinte au cas où la loi d'état—tout en généralisant légèrement la loi classique des gaz à chaleurs spécifiques constantes—conduit à des problèmes relatifs à une équation simple du type d'Euler-Poisson:

$$(r+s)t_{rs} - m(t_r + t_s) = 0;$$

les résultats classiques de Darboux concernant cette équation [Leçons sur la théorie générale des surfaces, vol. II, 2ème éd., Gauthier-Villars, Paris, 1915, livre IV, chap. III] trouvent ici de nombreuses applications, spécialement dans le cas où m est un entier négatif. Sont ainsi étudiés l'interaction de deux ondes simples pour un gaz diatomique, la détente dans le vide d'un gaz monoatomique et certains mouvements dans un tube fermé aux deux extrémités. On trouvera également, dans le cas général où la loi d'état n'est pas spécifiée, un exposé de la méthode de Riemann, de la construction pas à pas de la solution par la méthode des caractéristiques et des récentes recherches de Ludford sur l'extension de la méthode de Riemann à la définition de solutions sur des surfaces à plusieurs feuillets [Proc. Cambridge Philos. Soc. 48, 499–510 (1952); ces Rev. 13, 1001, 1140].

P. Germain (Providence, R. I.).

v. Krzywoblocki, M. Z. On vortex-equation in isentropic flow. J. Phys. Soc. Japan 8, 387–389 (1953).

Reprenant des raisonnements de Ringleb [Z. Angew. Math. Mech. 20, 185–198 (1940); ces Rev. 2, 169], l'auteur étudie, dans le plan de l'hodographe, les équations différentielles du tourbillon pour un écoulement gazeux, plan et adiabatique.

R. Gerber (Toulon).

Ovsyannikov, L. V. The equations of transsonic motion of a gas. Vestnik Leningrad Univ. 1952, no. 6, 47–54 (1952). (Russian)

By perturbing uniform sonic flow the author first derives approximate equations for steady plane irrotational flow originally obtained by von Kármán [J. Math. Phys. 26, 182–190 (1947); these Rev. 9, 217] and S. V. Falkovich [Akad. Nauk SSSR Prikl. Mat. Meh. 11, 459–464 (1947); these Rev. 9, 476]. Then he obtains appropriate forms for the conditions at a strong shock and shows that the same system of partial differential equations is valid to the same order of accuracy in transonic rotational flow behind a curved shock.

J. H. Giese (Havre de Grace, Md.).

Miles, John W. Impulsive motion of a flat plate. Quart. J. Mech. Appl. Math. 6, 129–140 (1953).

The aerodynamic problem is concerned with the transient transverse motion of a two-dimensional flat plate of finite width moving in inviscid compressible fluid. The initial and asymptotic relations responding to an abrupt transverse velocity of the plate are calculated and the force-time curve is fairly accurate over the entire range. The inverse problem, i.e., the velocity produced by a unit step force is also studied. It shows a cusp in acceleration at the time required for sound to travel twice the plate width ($t = 2$). No particular

physical explanation is given. The reviewer inclines to believe that cusps in acceleration might occur at $t=2n$ ($n=1, 2, 3\cdots$). As expected, the acceleration is nearly constant at large value of t. The above results should be restricted to transverse velocity much less than sound velocity. Schwinger's variation method is also applied to check the asymptotic behavior of the inverse case.
C. C. Chang (College Park, Md.).

Barish, David T., and Guderley, Gottfried. Asymptotic forms of shock waves in flows over symmetrical bodies at Mach 1. J. Aeronaut. Sci. 20, 491–499 (1953).

At unit Mach number of a free stream, no shock occurs in front of a symmetrical body either planar or axisymmetrical. No shock occurs immediately after the body in the downstream, if it is thin and smooth enough. However, a shock forms somewhere away from the body, owing to accumulation of compression waves propagated from the rear portion of the body. At large distances away from the body the authors show that similar solutions exist not only ahead but also after the downstream shock and are compatible with the shock relations. Essentially, a nonlinear ordinary differential equation of second order is solved graphically. Far away from the body the shape of the streamlines are obtained by integration. In the planar case the displacement of the streamline is proportional to the $\frac{1}{5}$ power of the vertical ordinate. This unexpected flow pattern is obscure although the authors give the reason that the streamlines are close together at sonic velocity. This shows that the flow field has a form which is independent of the body shape. There are some misprints in Figures 3 and 4 which can, however, be traced out of the table. References on the previous treatment mentioned are unfortunately missing.
C. C. Chang (College Park, Md.).

Ludloff, H. F., and Friedman, M. B. Mach reflexion of shocks at arbitrary incidence. J. Appl. Phys. 24, 1247–1248 (1953).

***Pack, D. C. Hodograph methods in gas dynamics.** The Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, College Park, Md., 1952. i+59 pp. (mimeographed)

Lecture notes based largely on the published work of Lighthill and Cherry. *J. B. Serrin* (Cambridge, Mass.).

Byrd, Paul F., und Huggins, Mary T. Zur Berechnung von Wirbelverteilung und Auftrieb eines dünnen Unterschallprofils in zwei hintereinander angeordneten Flügelgittern bei kompressiblen Strömungen. Ing.-Arch. 21, 191–193 (1953).

Following the work of Dörr [Ing.-Arch. 19, 66–68 (1951); these Rev. 13, 44] and of Nickel [ibid. 20, 6–7 (1952); these Rev. 14, 54] on thin air-foils in cascades, the present authors attack the problem of two unstaggered cascades in tandem. The cascades have equal blade spacings but arbitrary chords and axial spacing. The basic formula is the integral expression for the induced upwash on any blade. This can be inverted in closed form by a method used by Lomax and Byrd [NACA Tech. Note no. 2554 (1951)]; the result gives the circulation distributions. These are evaluated for the particular case of uncambered blades at incidence, and this result is used to write a simple formula for the total lift for any cambered blades, by means of the reverse-flow principle. It is shown how the same methods can be used for cascades of uncambered blades with small thickness.
W. R. Sears (Ithaca, N. Y.).

Lomax, Harvard, and Sluder, Loma. Chordwise and compressibility corrections to slender-wing theory. NACA Rep. no. 1105, ii+19 pp. (1952).

"Supersedes NACA Tech. Note no. 2295 (1951)" [these Rev. 13, 84].

Horton, C. W. On the diffraction of a plane sound wave by a paraboloid of revolution. II. J. Acoust. Soc. Amer. 25, 632–637 (1953).

In an earlier paper [same J. 22, 855–856 (1950); these Rev. 12, 650] the author and F. C. Karal, Jr. discussed the problem of the diffraction of a plane sound wave by the convex surface of a paraboloid of revolution. The solution was obtained in terms of paraboloidal wave functions. In the present paper the author has substantially advanced this problem as well as other paraboloidal wave problems by giving tables of the second solution $U_n^m(iy)$ to the Laguerre equation. Some asymptotic expressions for the paraboloidal wave functions are given. Two three-dimensional plots of these functions are also given. An expression for the pressure on the paraboloid is given. This is computed numerically and plotted for the region near the nose in one particular case. *E. Pinney* (Berkeley, Calif.).

Elasticity, Plasticity

Teissier du Cros, F. Les points singuliers du champ d'équilibre élastique à deux dimensions. Ann. Ponts Chaussées 122, 1–25 (1952).

As in a previous paper by the same author [C. R. Acad. Sci. Paris 233, 127–129 (1951); these Rev. 13, 300] the two-dimensional problem of plane strain or plane stress in an isotropic material is shown to be completely soluble in terms of two arbitrary complex functions. Two simple applications are given. The author then goes on to investigate the nature of the solution arising from values of the two complex functions where these functions have singularities. The functions which give rise to stresses which have a simple pole at a point are shown to correspond to a single concentrated force at that point, while the functions which give rise to stresses having a pole of order two, correspond to an isolated couple at the point. These results have been given by Stevenson [Proc. Roy. Soc. London. Ser. A. 184, 129–179, 218–229 (1945); these Rev. 8, 115] in terms of complex functions, and similar complex methods were also developed by Kolosoff and Muschelisvili [Ann. Inst. Electrotech. Petrograd 12, 39–55 (1915)].

The author does then treat the more general case when the stresses have poles of order k, and does say that since point loading cannot be achieved in practice, but must be spread over a finite area, such poles in the stresses may depend on the form of such a finite area.
R. M. Morris (Cardiff).

Ševčenko, K. The plane problem for an infinite elastic medium weakened by a circular cylindrical region. Doklady Akad. Nauk SSSR (N.S.) 89, 799–800 (1953). (Russian)

The author uses bipolar coordinates to solve the elastostatic problem stated in the title, or an equivalent problem of deformation of an infinite elastic plate with a circular hole. It is supposed that a pair of oppositely directed equal concentrated forces act at the extremities of a diameter of the hole and at infinity the plate is subjected to a uniform radial tension or compression. *I. S. Sokolnikoff*.

Sengupta, H. M. On the bending of an elastic plate. III. Bull. Calcutta Math. Soc. **44** (1952), 111–123 (1953).

[For parts I and II see same Bull. **41**, 163–172 (1949); **43**, 123–131 (1951); these Rev. **11**, 287; **13**, 1005.] The author determines the deflection of the central plane of a thin elliptic plate which is clamped at the edge and subject to a load uniformly distributed along a straight line parallel to the major axis. The problem is solved in elliptic coordinates and the solution is given as the sum of a singular term and an infinite series in products of trigonometric and hyperbolic functions. *J. L. Ericksen* (Washington, D. C.).

Ornstein, Wilhelm. Note on rectangular plates: deflection under pyramidal load. Quart. Appl. Math. **11**, 339–341 (1953).

Merlino, Francesco Savario. Contributo allo studio della lastra circolare su suolo elastico. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **14**, 231–237 (1953).

An infinite elastic plate is supported by an elastic foundation, and is subject to applied load. The solution of this problem (which is of considerable technical importance) has been discussed previously by several authors [e.g. see R. K. Livesley, Quart. J. Mech. Appl. Math. **6**, 32–44 (1953); these Rev. **14**, 819], and it is well-known that, subject to certain simplifying assumptions, the analysis of the statical problem involves the solution of the equation $D\nabla^4w+kw=q$ (the symbols having their usual meaning). The complete solution of this equation for arbitrary q has been given in simple form by M. Wyman [Canadian J. Research. Sect. A. **28**, 293–302 (1950); these Rev. **11**, 757]. The present author appears to be unacquainted with Wyman's work. He confines attention to certain particular cases of radial symmetry, and constructs solutions from the fundamental solution for a concentrated load through the use of reciprocal principles. *H. G. Hopkins* (Providence, R. I.).

Sokolov, S. N. Circular plate on a generalized elastic base. Akad. Nauk SSSR. Inženernyĭ Sbornik **11**, 161–168 (1952). (Russian)

A generalized base is a base, which gives not only a reactive force against displacements of an element of the plate, but also a reactive moment against a rotation of the element. So the reaction on the plate consists of a distributed pressure $p=p_a-kw$ (p_a and k are constants, w is the deflection of the plate) and a distributed moment, which in the case of axial symmetry is given by the expression $m=k_1dw/dr$ (k_1 is a constant). In this case, the deflection of the plate satisfies the differential equation

$$(1)\quad \frac{d^4w}{dx^4}+\frac{2}{x}\frac{d^3w}{dx^3}-\left(\frac{1}{x^2}+2C\right)\frac{d^2w}{dx^2}+\left(\frac{1}{x^3}-\frac{2C}{x}\right)\frac{dw}{dx}+w=\frac{p_a}{k},$$

where $x=r/\beta$, $\beta^4=D/k$, $C=\beta^2k_1/2D$ (D cylindrical stiffness of the plate). Equation (1) can be solved in terms of Bessel functions. The form of the solution depends on whether $C<1$, $C>1$, or $C=1$. *W. H. Muller* (Amsterdam).

Reckling, K. A. Die dünne Kreisplatte mit pulsierender Randbelastung in ihrer Mittelebene als Stabilitätsproblem. Ing.-Arch. **21**, 141–147 (1953).

General analysis is given of the transverse vibrations of a thin, elastic, circular plate induced by a pulsating force of type $a+b\cos\omega t$ applied at the edge and acting in the plane of the plate. Symmetric boundary conditions (including simple and built-in support) are assumed at the plate edge. Symmetrical and asymmetrical modes of vibration are studied, and the question of resonance (stability) is investigated. Some numerical results are given. *H. G. Hopkins* (Providence, R. I.).

Rohde, F. Virginia. Large deflections of a cantilever beam with uniformly distributed load. Quart. Appl. Math. **11**, 337–338 (1953).

Hatiašvili, G. M. On the deformation of a cylindrical composite beam with a loaded lateral surface. Soobščeniya Akad. Nauk Gruzin. SSR **13**, 335–341 (1952). (Russian)

The deformation of a homogeneous elastic beam when one of its bases is fixed and the other is free and when traction on the lateral surface is independent of the coordinate along the axis of the beam was solved by J. H. Michell and E. Almansi [A. E. H. Love, Mathematical theory of elasticity, 4th ed., Cambridge, 1927, pp. 348–359]. The author extends the solution of this problem to a compound beam whose cross-section consists of m regions S_i bounded by non-intersecting contours L_i contained within the region S_0 bounded by L_0. The contour L_0 corresponds to the trace of the lateral surface.

The elastic media filling the regions S_i ($i=0, 1, 2, \cdots, m$), may have distinct Young's moduli E_i, but the Poisson ratios are supposed to be the same throughout the cross-section. The components of the stress tensor are assumed to have special forms involving four unknown functions; three of these can be determined by solving certain two-dimensional Neumann problems in Laplace's equation and the fourth by solving the standard boundary-value problem in the biharmonic equation. The expressions for displacements are given in terms of these functions. *I. S. Sokolnikoff.*

Gorgidze, A. Ya. Stretching and bending by couples of naturally twisted composite beams. Soobščeniya Akad. Nauk Gruzin. SSR **13**, 73–80 (1952). (Russian)

Ruhadze, A. K. The problem of stretching of naturally twisted beams composed of various elastic materials. Soobščeniya Akad. Nauk Gruzin. SSR **13**, 137–144 (1952). (Russian)

Ruhadze, A. K. The problem of bending by couples of naturally twisted prismatic beams composed of various elastic materials. Soobščeniya Akad. Nauk Gruzin. SSR **13**, 265–272 (1952). (Russian)

Šarangiya, A. G. On bending by couples of a twisted beam composed of different materials. Soobščeniya Akad. Nauk Gruzin. SSR **13**, 389–396 (1952). (Russian)

The problems of extension, torsion and bending by end couples of prismatic beams composed of different elastic materials were formulated and solved under certain restrictions by N. I. Mushelišvili [C. R. Acad. Sci. Paris **194**, 1435–1437 (1932); Izvestia Akad. Nauk SSSR. Otd. Mat. Estest. Nauk (7) **1932**, 907–945]. A detailed account of the state of problems up to 1949 is contained in Mushelišvili's monograph, "Some fundamental problems of the mathematical theory of elasticity" [3rd ed., Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1949, pp. 538–591; these Rev. **11**, 626].

The object of the papers under review is to extend the Mushelišvili solutions to composite initially twisted beams. The cross-section S of the beam is assumed to consist of several closed nonintersecting simply connected domains S_i ($i=1, 2, \cdots, m$), contained within a closed region S_0

representing the elastic medium surrounding the domains S_i. The domains S_i represent the cross-sections of reinforcing rods with different elastic properties.

In the papers by Gorgidze and Ruhadze it is assumed that the cross-sections of the beam in the initial unstressed state are rotated so that the natural angle of twist $\theta(z)=kz$, where z is measured along the length of the beam and k is a constant so small that its second and higher powers can be disregarded in computations. It is supposed that the lateral surface of the beam is free of tractions, the components of displacements are continuous throughout S, and the stress vectors acting on surfaces separating the regions with different elastic properties are equal in magnitude and are opposite in direction.

Gorgidze solves the problem of simple extension and pure bending by supposing that the Young moduli E_i in each region S_i are different but the Poisson ratios σ_i are all alike. The undesirable restriction on the σ_i is removed in the two papers by Ruhadze, who finds it necessary to solve an auxiliary problem in plane elasticity in order to satisfy the condition of the continuity of displacements throughout the region S. The hypothesis of the equality of the σ_i, made by Gorgidze, was introduced to ensure the continuity of displacements in his solutions.

Šarangiya's paper contains a solution of the problem of bending by end couples of an initially straight composite beam which is twisted by end couples with moments along the axis of the beam. The cross-section S of the beam has the form described above. The problem of pure bending of naturally twisted but initially unstressed beams has many points of similarity with that of pure bending of initially torsioned beams. To ensure the continuity of displacements in S, when the Young moduli and Poisson's ratios are distinct in the domains S_i, Šarangiya has to solve an auxiliary plane problem identical to that formulated by Ruhadze. The problem of pure bending of an initially torsioned homogeneous beam was first solved by Gorgidze and Ruhadze [Soobščeniya Akad. Nauk Gruzin. SSR 5, 253–262 (1944); these Rev. 7, 230]. *I. S. Sokolnikoff.*

Lur'e, A. I. Equilibrium of an elastic hollow sphere. Akad. Nauk SSSR. Prikl. Mat. Meh. 17, 311–332 (1953). (Russian)

The equation of equilibrium, of isotropic elastic material under no body forces, in terms of the displacement $\mathbf{u}$ is satisfied by

$$\mathbf{u}=\frac{4(m-1)}{m}\mathbf{B}-\text{grad }(\mathbf{R}\cdot\mathbf{B})-\text{grad }B^0,$$

where $1/m$ is Poisson's ratio, $\mathbf{R}$ is the position vector, $\mathbf{B}$ is a vector harmonic function and B^0 is a scalar harmonic function. This expression, attributed to Papkovič (no reference given), forms the author's starting point for the solution, using vector methods exclusively, of the following problems concerning a sphere with a concentric spherical hole: (i) given displacements on the boundary, (ii) given surface tractions, (iii) given equilibrating concentrated forces. The author uses essentially the method given by Kelvin for solving (i) [Thomson and Tait, Treatise on natural philosophy, v. I, new ed., Cambridge, 1883]. Problem (ii) in the particular case of axial symmetry was solved by B. G. Galerkin [Akad. Nauk SSSR. Prikl. Mat. Meh. 6, 487–496 (1942); these Rev. 5, 27]. A solution of problem (iii) was offered by C. Weber for radial forces [Z. Angew. Math. Mech. 32, 186–195 (1952); these Rev. 14, 335]. The author's general solutions of (ii) and (iii) appear to be new.

L. M. Milne-Thomson (Greenwich).

Ziegler, Hans. Linear elastic stability. A critical analysis of methods. Z. Angew. Math. Physik 4, 89–121, 167–185 (1953).

Die charakteristische Situation bei Stabilitätsproblemen wie z.B. Knickproblemen, kritischen Drehzahlen usw. ist die folgende: Bei genügend kleinen Lasten gibt es eine Konfiguration q des stabilen Gleichgewichtes in der unmittelbaren Nachbarschaft der Gleichgewichtslage q_0 des unbelasteten Systems. Bei gewissen kritischen Lasten wird q entweder unstabil oder so weit von q_0 enfernt, dass das System in eine kritische Lage gebracht wird. Die verschiedenen Lösungsmethoden sind bisher nie verglichen oder gar kritisch analysiert, obwohl ihre Beziehungen nur lose und in keiner Weise einleuchtend sind. Darum kann es vorkommen, dass eine Anzahl von offenbar einfachen Problemen, nach verschiedenen Methoden gelöst, zu verschiedenen in einigen Fällen sogar zu ganz unwahrscheinlichen Ergebnissen führen. Die Ursache zu diesen Widersprüchen sind in dem Umstand zu suchen, dass die betrachteten Systeme nicht konservativ sind. Bei Verbesserung der Grundannahmen ist est möglich, zuverlässige Ergebnisse zu erhalten. Dieser Sachverhalt zeigt das Bedürfnis nach einer Analyse der Stabilitätsprobleme sowie der Verfahren zu deren Lösung. Die vorliegende Arbeit bringt auf Grund einer zweckmässigen Einteilung der verschiedenen mechanischen Systeme eine solche Analyse. Um unnötige Komplikationen zu vermeiden, ist die Untersuchung auf holonome und skleronome Systeme mit Differentialgleichungen beschränkt, die linearisiert werden können. Das Ergebnis der Untersuchung zeigt, dass elastische Systeme im wesentlichen in konservative, nämlich gyroskopische und nichtgyroskopische sowie nichtkonservative, nämlich dissipative, zirkulatorische und instationäre Systeme eingeteilt werden können. Bei entsprechender Analyse der fünf verschiedenen Systeme stellt sich heraus, dass die bei Berechnung der Stabilität üblichen statischen Verfahren nur bei nichtgyroskopischen und mit gewissen Einschränkungen bei dissipativen Systemen angewandt werden können. Alle übrigen Systeme (gyroskopische, zirkulatorische sowie instationäre) müssen kinetisch behandelt werden.

R. Gran Olsson (Trondheim).

Gol'denblat, I. I. Dynamic longitudinal stability of thin-walled beams. Akad. Nauk SSSR. Inženernyĭ Sbornik 5, no. 1, 133–139 (1948). (Russian)

A thin-walled beam is loaded with a periodically variable thrust and can perform bending vibrations in two directions and torsional vibrations. In the simple case, when the cross-section of the bar has two axes of symmetry and the thrust is applied in the centre, the equation of motion is Hill's equation. The article also deals with the more complicated case, when the cross-section has only one axis of symmetry. Then the equations of motion are two simultaneous differential equations with periodical coefficients. The stability of the solutions of these equations is examined by extending the methods used in the corresponding theory of Hill's equation. *W. H. Muller* (Amsterdam).

Nikolenko, G. I. Vibrations of initially stressed elastic systems. Akad. Nauk SSSR. Inženernyĭ Sbornik 11, 79–94 (1952). (Russian)

A rigid body is suspended by n springs. The unloaded length of the kth spring is l_{k0}; when the body is in equilibrium under gravity, its length is l_k^0; and in an arbitrary position during the vibration it is L_k. When the stiffness of

the spring is C_k, the potential energy is $\Pi=\frac{1}{2}\sum_{i=1}^{n}C_k(L_k-l_{k0})^2$. By splitting up $L_k-l_{k0}=(L_k-l_k^0)+(l_k^0-l_{k0})$, the dynamic deflections $L_k-l_k^0$ and static deflections $l_k^0-l_{k0}$ are introduced separately. The dynamic deflections can be expressed in the displacements u, v, w, along three mutually orthogonal axes and the rotations $\theta_x, \theta_y, \theta_z$ about these axes. The potential energy becomes, with the usual omissions, a quadratic function of $u, v, w, \theta_x, \theta_y, \theta_z$ with coefficients depending on the static deflections. The latter are calculated from the equilibrium conditions of the body under its own weight. So the natural frequencies and modes, calculated with the aid of this expression for the potential energy, depend on the initially static stresses of the springs.

W. H. Muller (Amsterdam).

Jobert, N. Dispersion des ondes de Rayleigh en milieu hétérogène. Application au neve du Groenland. Ann. Géophysique 9, 28–32 (1953). (Esperanto summary)

The author investigates the dispersion of Rayleigh waves for the case of heterogeneous medium. The elastic properties of the medium are assumed to vary with depth as follows:

$$(\rho/\mu)=L+M\exp(-2kz),\quad \lambda=\lambda_\infty\exp(-B\exp(-kz)),$$

where L, M, λ_∞, B and k are constants; Poisson's ratio is assumed to be independent of depth. The limiting form of the dispersion relation for long waves is obtained (terms beyond the quadratic in p and f have been neglected in the calculation). The results have been applied to the case of propagation of Rayleigh waves in the top layer of the glacier in Greenland. *J. Shmoys* (New York, N. Y.).

Prager, William. A geometrical discussion of the slip line field in plane plastic flow. Trans. Roy. Inst. Tech. Stockholm no. 65, 27 pp. (1953).

In most of the solved problems in plane flow of a perfectly plastic material for arbitrarily shaped boundaries, it is assumed that the stresses are known along the boundary, and hence the stress field is statically determinate. Actually, the boundary conditions involve both stresses and velocities. In solving such mixed boundary-value problems, most authors proceed by intuition, assuming that one family of slip lines forms a fan, etc. The purpose of the present article is to determine geometric (and hence graphical) methods for determining the slip lines for arbitrary contours, in which mixed boundary values are given.

First, the author discusses the geometric relations between (1) the physical plane and the stress plane, and (2) the physical plane and the hodograph plane. In the discussion of the first topic, the properties of the Mohr circle are investigated. Here, geometric methods are given for determining the maps of the slip lines in the stress plane. These maps are a family of cycloids [R. Sauer, Z. Angew. Math. Mech. 29, 274–279 (1949); these Rev. 11, 284]. Similarly, the geometry of the slip lines in the hodograph plane is investigated.

Finally, the author considers the geometric construction of the slip lines for the following three types of boundary-value problems: (1) stresses are given along an arc (not a slip line) of a boundary; (2) two intersecting slip lines and the normal component of the velocity along these lines are given; (3) a single slip line, along which the normal component of velocity, and an arc, along which the slope of the slip lines, are given. A discussion of degenerate mappings (as in fans) and an example, involving inverted sheet extrusion, conclude the paper. *N. Coburn.*

Richter, Hans. Elasto-plastische Reflexion eines Stabes. Z. Angew. Math. Mech. 33, 237–244 (1953). (English, French and Russian summaries)

If a bar, finite in length, made of elastic-perfectly plastic material impinges normally against a rigid barrier, then, provided that the impact velocity is sufficiently high, some plastic deformation will occur at least during the time of impact. Analysis, based upon the theory of characteristics, is given of longitudinal wave propagation in a bar under these conditions. The dependence of the coefficient of restitution and impact time upon the length and the impact velocity of the bar is exhibited graphically.

H. G. Hopkins (Providence, R. I.).

Leibfried, G. Versetzungen in anisotropem Material. Z. Physik 135, 23–43 (1953).

Abgesehen von der Behandlung einiger ganz spezieller Fälle, liegen bis jetzt Berechnungen des elastischen Verschiebungsfeldes und der damit zusammenhängenden Grössen nur für isotrope Medien vor. Da andererseits solche Eigenschaften der Versetzungen wie Gleitebene und Gleitrichtung durch die atomistische Struktur bestimmt werden, so erweitert der Verfasser die erwähnten Berechnungen auf anisotrope Medien.

Nach einer Übersicht der Grundlagen der Elastizitätstheorie werden die Abweichungen vom isotropen Verhalten in erster Näherung berücksichtigt, in dem der elastische Tensor T_{nm}^{ik} in einem gemittelten (isotropen) Teil $\bar{T}$ und einem anisotropen δT zerlegt wird. Weiter wird die Bezeichnung $t_{in}(\mathfrak{k})=T_{nm}^{ik}k_mk_k$ eingeführt; für die reziproke dieser Matrix folgt dann in erster Näherung

$$t_{nl}^{-1}=\bar{t}_{nl}^{-1}-\bar{t}_{ns}^{-1}\delta t_{sr}\bar{t}_{rl}^{-1},$$

wo $\delta t_{nl}=\delta T_{lq}^{np}k_pk_q$ ist. t_{nl}^{-1} ist gleich der Fourier-Transformierten $\bar{S}_{nl}$ der Verschiebungen. Die erwähnte erste Näherung gestattet die Anteile eines Linienelementes einer Versetzungslinie an den Verschiebungen oder Spannungen durch elementare Funktionen auszudrücken. Im allgemeinen Fall wäre das nicht möglich. Diskutiert werden Stufen- und Schraubenversetzungen in kubischen und hexagonalen Kristallen, die ja bezüglich der Metalle allein von Bedeutung sind. Im Anhang werden erhaltene Formeln und numerische Ergebnisse tabellarisch zusammengestellt.

Th. Neugebauer (Budapest).

MATHEMATICAL PHYSICS

Electromagnetic Theory

Gercenštein, M. E. Scattering of radio waves by local nonuniformities of the ionospheric plasma. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. 23, 678–681 (1952). (Russian)

In a previous paper [same Žurnal 22, 303–309 (1952); these Rev. 14, 226] the author showed that if the effect of thermal motion of the electrons is taken into account, in addition to the transverse (radio) waves there exist longitudinal waves. The latter are attenuated even in absence of collisions between electrons and molecules. Inhomogeneities in the plasma cause attenuation of radio waves by scattering the radio wave directly and by converting it into a longitudinal wave. The two scattering cross-sections (for Z excess

electrons in an inhomogeneity) are calculated in the case of inhomogeneity much smaller in extent than the wavelength of the longitudinal waves; they turn out to be

$$\sigma_{\text{transv.}}=\frac{8\pi Z^2}{3}\left[1-\left(\frac{\omega_0}{\omega}\right)^2\right]^{-1},\quad \sigma_{\text{long.}}=\frac{8\pi Z^2}{6}\left(\frac{c}{v_0}\right)^2$$

in terms of the plasma frequency ω_0 and root mean square thermal velocity v_0. *J. Shmoys* (New York, N. Y.).

Aymerich, Giuseppe. Sulle onde elettromagnetiche guidate da una superficie cilindrica perfettamente conduttrice anisotropa. Rend. Sem. Mat. Univ. Padova **22**, 157–176 (1953).

This paper deals with the propagation of electromagnetic waves along a perfectly-conducting anisotropic cylindrical wave guide of arbitrary cross-section. The anisotropic surface of the guide can be regarded as the limiting form of a helical sheath built up out of a large number of perfectly conducting insulated wires, each in the form of a helix; on such a surface, the electric current at any point is always in the direction of the tangent to the helix through the point.
E. T. Copson (St. Andrews).

Sayasov, Yu. S. The phenomenon of a strong disturbance of characteristic electromagnetic oscillations in cylindrical regions for an insignificant violation of the cylindricity. Doklady Akad. Nauk SSSR (N.S.) **90**, 163–166 (1953). (Russian)

The author discusses the influence on the characteristic electromagnetic oscillations of small deformations of the side surfaces of long, perfectly conducting, cylindrical resonators. He asserts that the perturbation of the field is proportional not only to the relative deformation of the side surfaces, but also to an additional factor which depends on the square of the ratio of the length of the cylinder to its transverse dimension; if this ratio is large, a small deformation leads to a large perturbation field.

To support this assertion he examines in detail a special case where the unperturbed region is a circular cylinder and the perturbed region is formed by circumscribing the cylinder with a conical surface which passes through one of the end circles of the cylinder and with two spherical surfaces which pass through the two end circles of the cylinder. The surfaces which enclose the unperturbed and perturbed regions are the "natural" surfaces of cylindrical and spherical coordinate systems respectively, and, therefore, the field components of both regions are exactly represented in terms of well-known solutions of Maxwell's equations.

For this special case the author's assertion is indeed true, but one must proceed with caution in drawing generalizations regarding small arbitrary deformations.
C. H. Papas (Pasadena, Calif.).

De Socio, Marialuisa. Alcuni teoremi di unicità per le equazioni di Maxwell. Boll. Un. Mat. Ital. (3) **8**, 196–200 (1953).

The first uniqueness theorem proved here is that if the medium in a domain D, bounded by a surface σ, is homogeneous, then an electromagnetic field, regular and "monochromatic" in D, is uniquely determined by a knowledge of the tangential components of the electric and magnetic forces on a portion σ_1 of σ. This result is used to extend a uniqueness theorem for wave guides, due to Graffi [Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) **8**, 213–218 (1952); these Rev. **14**, 115] so as to cover the case when the walls of the guide are not perfectly conducting. *E. T. Copson.*

Quantum Mechanics

✶Barriol, Jean. Mécanique quantique. Presses Universitaires de France, Paris, 1952. xii+257 pp. 1,500 francs.

This excellent little book originated in lectures (given in the prison camps of Lübeck, Fischbach and Edelbach) introductory to the basic treatises of von Neumann and Dirac on quantum mechanics. It provides the best approach to these authors, difficult enough under the most favorable circumstances, with which the reviewer is familiar. Barriol's simple and elegant book is motivated by a group-theoretical approach, though no explicit use is made of representation theory. The use of the two-dimensional oscillator to illustrate degeneracy and its relation to the symmetry of the system is a particularly happy pedagogic device. The discussions of the consequences of $AB-BA=i$, of spin, of the Pauli Principle, of approximate wave functions for many-electron atoms are all extraordinarily lucid.
A. J. Coleman (Toronto, Ont.).

✶Flügge, Siegfried, und Marschall, Hans. Rechenmethoden der Quantentheorie dargestellt in Aufgaben und Lösungen. Erster Teil. Elementare Quantenmechanik. 2te Aufl. Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Band LIII. Springer Verlag, Berlin-Göttingen-Heidelberg, 1952. viii+272 pp. DM 29.80; bound, DM 32.80.

In revising the first edition [these Rev. **9**, 553] the authors attempted primarily to stress more the physical than the mathematical aspects of quantum theory. A mathematical appendix has been omitted. Of the 79 problems of the first edition 17 problems on relativistic quantum mechanics have been left aside for a planned second volume; instead, 43 new problems have been added. Many of these new problems are based on the use of methods which have not been included in the first edition: non-relativistic treatment of angular momentum and spin (8), perturbation and variation methods (18), and the treatment of many-body problems, in particular, of the electron gas in its ground state (10). This made it possible to enlarge the physical subject matter; there are now problems on nuclear forces, large atoms, magnetism, field emission, and astrophysics. These revisions make the book much more useful for both students and teachers. *E. Gora* (Providence, R. I.).

Fényes, Imre. Eine wahrscheinlichkeitstheoretische Begründung und Interpretation der Quantenmechanik. Z. Physik **132**, 81–106 (1952).

This paper belongs to the many recent attempts to line up the fundamental equations of quantum mechanics with those of classical stochastic processes, with the object of narrowing the gap between quantum and classical conceptualisms. The starting point of the paper is a continuous homogeneous Markoff process for Brownian movement, together with its Kolmogoroff and Fokker parabolic differential equations for transition probabilities and the state-probability densities. Attention is then restricted to the case of (expected-) velocity potentials. By introducing a "probability amplitude" Ψ in terms of the potentials, so that the state probability density is given by $|\Psi|^2$, the Fokker equation is made to assume the form of the equation of continuity (conservation of probability) derived from the Schrödinger equation. Further, by introducing a linear operator having a simple formal statistical meaning in the stochastic equations, commutator relations are found which

coincide with those underlying the "principle of indeterminacy" of quantum mechanics. These also are shown to have a physical meaning in the classical application to diffusion.

The author holds that this shows that "indeterminacy" is, even in quantum mechanics, at bottom a phenomenon of classical probability. He goes on to claim that causality can be restored in quantum mechanics by introducing "concealed parameters" by a more general method than that shown to be impossible by von Neumann. Much is made, in the course of the argument, of the notion that a single experiment cannot change a probability distribution, such as that contained in a wave function or an operator. To the reviewer, the arguments lack the precise and explicit character needed to bring conviction of the validity of the author's thesis. *B. O. Koopman* (New York, N. Y.).

Epstein, Saul T. The causal interpretation of quantum mechanics. Physical Rev. (2) **91**, 985 (1953).

Dubarle, D. Mécanique quantique et information. Rev. Questions Sci. (5) **14**, 347–368 (1953).

This paper is apparently not intended as a contribution to the physics or the mathematics of quantum mechanics, but rather to its conceptual background. The leading notion is that the result of an observation can not only be regarded as an initial wave function from which later states can be computed by the Schroedinger equation, but equally well as a final state from which an initial one can be computed by means of a sort of dual analogue of Schroedinger's equation. The construction of the latter equation is sketched out discursively in the paper, but no definitive rule for finding it in all classical cases is provided. By thinking of a pair of observations as yielding information of both past and future, the notion of the possibility of a more complete datum concerning the state is developed, which might remove the usual lack of causality. The level of such a description would appear to be philosophical rather than experimental. As little in the way of precise mathematical development is given, it is hardly possible to evaluate the final contribution of the paper. *B. O. Koopman* (New York, N. Y.).

Husimi, Kôdi. Miscellanea in elementary quantum mechanics. II. Progress Theoret. Physics **9**, 381–402 (1953).

[For part I see same journal **9**, 238–244 (1953); these Rev. **14**, 1047.] A detailed treatment of forced harmonic oscillations in quantum mechanics, both the impressed force and the natural frequency being given functions of time. Transition probabilities are computed and the classical limit discussed. The symmetry of the transition probabilities with respect to initial and final states is analysed and the results shown to be identical with those derived by modern perturbation theory. *H. C. Corben* (Pittsburgh, Pa.).

Brdička, M. A remark on proper Lorentz transformation of Dirac's equations. Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **51** (1950), 101–108 (1953).

A Czech version of this paper was reviewed earlier [Rozpravy II. Třídy České Akad. **60**, no. 12 (1950); these Rev. **12**, 887].

Costa de Beauregard, Olivier. Une réponse à l'argument dirigé par Einstein, Podolsky et Rosen contre l'interprétation bohrienne des phénomènes quantiques. C. R. Acad. Sci. Paris **236**, 1632–1634 (1953).

This refers to a paper by Einstein, Podolsky and Rosen [Physical Rev. (2) **47**, 777–780 (1935)], where an example is given in which two physical variables, non-commuting according to quantum mechanics, can be alternatively determined by means of measurements carried out at a distance from the system being measured. In the present paper the author discusses another example in which one can determine two such physical quantities alternatively by making adjustments in the apparatus after the interaction between the measuring instrument and the observed system has taken place. *N. Rosen* (Haifa).

Demkov, Yu. N. Variational principles and the virial theorem for problems of the continuous spectrum in quantum mechanics. Doklady Akad. Nauk SSSR (N.S.) **89**, 249–252 (1953). (Russian)

The variational principle for the continuous spectrum of the Schrödinger equation in a central potential, first proposed by L. Hulthén [Kungl. Fysiografiska Sällskapets i Lund Forhandlingar [Proc. Roy. Physiog. Soc. Lund] **14**, no. 21 (1944); these Rev. **6**, 111], is recalled and applied to the variation of the wave function produced by a change in the unit of length. The result is the expression for the derivative of the phase shifts with respect to the wave number in terms of the wave function, the potential and the force. The same method, applied to the discrete spectrum, gives Fock's derivation of the quantum-mechanical virial theorem [V. Fock, Z. Physik **63**, 855–858 (1930)]. The extension to a non-central potential is given. It is based on W. Kohn's form of the variational principle [Kohn, Physical Rev. (2) **74**, 1763–1772 (1948)]. *L. Van Hove.*

Källén, Gunnar. On the magnitude of the renormalization constants in quantum electrodynamics. Danske Vid. Selsk. Mat.-Fys. Medd. **12**, 18 pp. (1953).

This important paper settles what has been for many years one of the outstanding open questions in physics. Namely, are the divergences of quantum electrodynamics inherent in the theory, or are they introduced by the use of perturbation treatments which expand all physical quantities in powers of the fine-structure constant α? The author shows that the first alternative is the correct one.

The central difficulty in settling this question has been the following. The only known solutions of the equations of quantum electrodynamics are formal power series in α obtained by a perturbation treatment. The coefficients in these series contain divergences. But as no other solutions of the equations are known, it seemed impossible to decide whether solutions of a different type might be free of divergences. The author has overcome this difficulty by constructing a new logical basis for the whole theory of quantum electrodynamics. He takes as starting point the postulate that there exists a certain set of quantum-mechanical operators $\psi_\alpha(x)$, $A_\mu(x)$ which satisfy the requirements of quantum mechanics and the field equations of the theory. He does not assume that he knows how to construct these operators, or that they are expressible as power series in α. Nevertheless, he is able to deduce a mathematical contradiction from the hypothesis that such a set of operators forms a divergence-free system, and so he settles the question stated above.

In an earlier paper [Helvetica Phys. Acta **25**, 417–434 (1952); these Rev. **14**, 435] the author reconstructed the method of mass- and charge-renormalization in quantum electrodynamics on the basis of his new postulate, so that the renormalization was defined independently of a series expansion in α. As the earlier paper was not reviewed in detail, this review will cover both papers. He started from a formalism in which the field-operators $\psi_\alpha(x)$ and $A_\mu(x)$ are

taken to be "renormalized" quantities, i.e., they have finite matrix elements between experimentally realizable states. Also m and e are the finite experimental mass and charge of the electron. Then the field equations from which the theory starts are

$$(1)\quad [\gamma_\nu(\partial/\partial x_\nu)+m]\psi(x)=\tfrac{1}{2}ie\{A_\nu(x),\gamma_\nu\psi(x)\}+K\psi(x),$$

$$(2)\quad \Box A_\mu(x)=-\tfrac{1}{2}ieN^2[\bar{\psi}(x),\gamma_\mu\psi(x)] +L(\Box A_\mu(x)-(\partial^2 A_\nu(x)/\partial x_\mu\partial x_\nu)).$$

Here K, L, N^2 are the "renormalization constants" for the mass, charge, and probability density of the electron respectively. When solutions are constructed by series expansion in α, it is known that equations (1), (2) lead to finite expressions for $\psi(x)$, $A(x)$ in which the constants K, L, N drop out. The constants K, L, N, which are unobservable, must be equated formally to power series in α all the coefficients of which are divergent integrals. In this way, within the framework of the series expansion, the divergences appear in the starting equations but not in the observable consequences of the theory.

In the paper cited, the author showed that equations (1), (2) may still be used as the starting point of the theory, when no expansion in α is made. In this case we cannot construct solutions, but we can postulate that solutions exist and examine their formal properties. The author showed that equations (1), (2) were mathematically consistent provided that K, L, N were equated to certain closed expressions involving matrix elements of the postulated operators ψ_α, A_μ. For example, L is given by

$$(3)\quad \frac{L}{1-L}=\int_0^\infty \pi(-x)\frac{dx}{x},$$

$$(4)\quad \pi(p^2)=-\frac{V}{3p^2}\sum_z\langle 0|j_\nu|z\rangle\langle z|j_\nu|0\rangle,$$

the summation being taken over all states z of momentum p, and $\langle z|j_\nu|0\rangle$ being the matrix element of the current operator j_ν between the vacuum state 0 and z. The function $\pi(p^2)$ is finite for all p, because the states z in (4) extend over only a finite volume of phase-space. The theory will be completely divergence-free only if integrals such as (3) converge at infinity. On the other hand, the theory will be mathematically consistent and all physically observable quantities will be finite if integrals such as

$$(5)\quad \int_0^\infty \pi(-x)\frac{dx}{x^2}$$

converge at infinity.

In the paper under review, using equations (3), (4) and their analogs for the constants K, N, the author shows that not all of K, N, L can be divergence-free. The decisive weapon is the fact that $\pi(p^2)$ defined by (4) is positive-definite. Assuming N and L finite, it is shown that $\pi(p^2)$ is for sufficiently large p not less than a finite number (depending on N and L), and so (3) diverges logarithmically just as it did in the series expansion solution. The proof proceeds by a very ingenious use of the field equations (1), (2), combined with the assumption that a complete set of quantum-states z exists for every momentum p.

This analysis leaves open the even more difficult question whether a physically admissible solution of the field equations exists. If it could be shown that one of the integrals such as (5) necessarily diverges, the question would be answered in the negative. In the more likely event that the convergence of (5) and similar integrals is formally consistent, no conclusion could be drawn concerning the existence of solutions. However, the author's analysis has greatly clarified the nature of the basic problem we are facing in quantum field theory, which is to discover the mathematical nature of the objects we are handling. *F. J. Dyson.*

Friedrichs, K. O. Mathematical aspects of the quantum theory of fields. IV. Occupation number representation and fields of different kinds. Comm. Pure Appl. Math. 5, 349–411 (1952).

This installment follows three earlier papers of the author [same Comm. 4, 161–224 (1951); 5, 1–56 (1952); these Rev. 13, 520, 894]. He here analyses in meticulous detail and with much new terminology the mathematical definition of the "occupation-number representation" for a linear quantized field without interaction. He finds appropriate notations for the wave-function in the occupation-number representation, when the states of a single quantum form a continuous rather than a discrete manifold. The wave-function is defined by a limiting process which seems to the reviewer not more rigorous than the limiting process, customarily used by physicists, in which the quantum states are made discrete by enclosing the universe in a finite box.

The second half of this paper deals with what the author calls "myriotic" and "amyriotic" fields. A "myriotic" field is one which gives zero probability for finding any finite number of quanta present. An example familiar to physicists is the electromagnetic field in the neighborhood of an electric charge. An "amyriotic" field is one which is not myriotic. The author shows how wave-functions, expectation values, etc. are to be defined for myriotic fields. His procedures do not differ much, except in nomenclature, from the methods used by F. Bloch and A. Nordsieck [Physical Rev. (2) 52, 54–59 (1937)] in their original discussion of the electromagnetic field radiated by a moving charge.

F. J. Dyson (Princeton, N. J.).

Friedrichs, K. O. Mathematical aspects of the quantum theory of fields. V. Fields modified by linear homogeneous forces. Comm. Pure Appl. Math. 6, 1–72 (1953).

Continuing a series of papers [see the preceding review], this paper discusses the mathematics of a linear field disturbed by forces which are themselves linear in the field amplitudes. The most important example of such a field is the theory of a collection of non-interacting electrons each moving independently in a given classical electric field. Since the total field equations are linear and homogeneous, the stationary states of the quantized theory can be expressed exactly in an occupation-number representation, provided a complete set of eigenfunctions exists for the unquantized field equations. The whole paper is an elaboration of the formal connections between the wave-functions described in terms of "bound-particle" and "free-particle" occupation numbers. There is no discussion of the unsolved mathematical problem, to find the conditions to be satisfied by the linear forces in order that a complete set of eigenfunctions should exist; therefore the scope of the analysis remains somewhat undefined. *F. J. Dyson.*

4/ ***Friedrichs, K. O. Mathematical aspects of the quantum theory of fields.** Interscience Publishers, Inc., New York; Interscience Publishers Ltd., London, 1953. viii+272 pp. $5.00.

Reprinted from Comm. Pure Appl. Math. 4, 161–224 (1951); 5, 1–56, 349–411 (1952); 6, 1–72 (1953) [these Rev.

13, 520, 894, and the two preceding reviews]. Pages 257–272 contain an appendix and additional comments and corrections to the original five parts.

Schwinger, Julian. The theory of quantized fields. II. Physical Rev. (2) 91, 713–728 (1953).

This paper is partly a revised version, and partly an extension, of the formal theory of quantized fields presented in the author's earlier paper [Physical Rev. (2) 82, 914–927 (1951); these Rev. 13, 520]. The aim is to derive from the fewest and simplest hypotheses the complicated and apparently arbitrary assumptions upon which the quantum field theories now in current use are based. The author succeeds, for example, in rationalizing the division of all field quantities into commuting and anti-commuting types, deducing this division from the fact that the coefficient matrix of a quadratic Lagrangian divides into a symmetric and an antisymmetric part. The last section of the paper deals with the special properties of the electromagnetic field, which appear as necessary consequences of the existence of a gauge-group under which the formalism is invariant. The details of the author's arguments are too involved to be usefully summarized. *F. J. Dyson* (Princeton, N. J.).

Schwinger, Julian. The theory of quantized fields. III. Physical Rev. (2) 91, 728–740 (1953).

This paper is an exercise in applying the author's general theory of quantized fields [see the preceding review] to a simple example. The example chosen is the theory of the electromagnetic field under the influence of an external classical current distribution. This problem can be solved exactly, and all transition matrices, eigenvalues, and eigenvectors can be calculated explicitly. The author carries through these calculations in detail, obtaining results in agreement with more elementary treatments.

F. J. Dyson (Princeton, N. J.).

Pauli, W. On the Hamiltonian structure of non-local field theories. Nuovo Cimento (9) 10, 648–667 (1953).

The author considers the class of field theories with non-local interactions, of which the theory of P. Kristensen and C. Møller [Danske Vid. Selsk. Mat.-Fys. Medd. 27, no. 7 (1952)] is a typical example. The Kristensen-Møller theory has two fields $u(x)$ and $\psi(x)$, satisfying field equations of the form

$$(1)\qquad (\Box - m^2)u(x) = g\int\int F(x', x, x'')\bar{\psi}(x')\psi(x'')dx'dx'',$$

where F is a relativistically invariant function whose exact form is unspecified. The meaning of the theory is obscured by the fact that the right side of (1) contains field-quantities at times both earlier and later than the time of x. Therefore, the theory does not, like ordinary mechanics, determine the time-derivatives of the field-quantities in terms of their values at a given time.

This paper is concerned with the question, how far such a non-local theory can be brought into correspondence with the formal structure of Hamiltonian dynamics, so that standard methods of analytical mechanics may be applied to it. It is found that there is a surprisingly wide range of concepts in the Hamiltonian formalism, which can be extended to the non-local theory. The author's general conclusion is that local and non-local theories are not necessarily different in their causal structure, so long as their properties are considered only in one Lorentz frame of reference. He considers the fundamental difference between them to lie in the fact that the canonical field variables of the non-local theory do not transform under Lorentz transformations in any simple way. However, the behavior of the non-local theory under Lorentz transformations is not investigated in this paper.

In the first section it is shown that exact differential and integral conservation laws for charge, momentum, and energy exist in the non-local theory, and explicit formulae are given for all these quantities. This contradicts a prevalent opinion that only asymptotic conservation laws should exist in such cases.

Section 2 begins with a new general theorem in the dynamics of systems with a finite number of degrees of freedom. Suppose that for any two physical quantities A, B a Poisson bracket $[A, B] = -[B, A]$ can be defined, satisfying the usual distributive laws and also the Jacobi identity

$$(2)\qquad [A, [B, C]]+[B, [C, A]]+[C, [A, B]]=0.$$

Then there exist variables p_α, q_α which possesses the canonical Poisson brackets

$$(3)\qquad [q_\alpha, q_\beta]=[p_\alpha, p_\beta]=0,\quad [p_\alpha, q_\beta]=-i\delta_{\alpha\beta},$$

and in terms of which the equations of motion take the canonical Hamiltonian form. The author assumes this theorem to hold also for field theories which have an infinite number of degrees of freedom, suitable changes being made in the notations.

Using the theorem, in section 3 the author defines Poisson brackets and proves the existence of canonical variables, for every classical non-local field theory belonging to the "normal class". The normal class of theories are those in which a given set of values of the field quantities at a given time determines the past and future motion uniquely.

In section 4 the canonical variables of the quantized non-local theory are calculated by perturbation theory to first order in the coupling constant g. The author believes that canonical variables will always exist in quantized theories of the normal class, but this conjecture is still unproved.

F. J. Dyson (Princeton, N. J.).

Chrétien, M., and Peierls, R. E. Properties of form factors in non-local theories. Nuovo Cimento (9) 10, 668–676 (1953).

Let $F(x)$ be a relativistically invariant function of the 4-vector x, which tends to zero rapidly for values of $|x^2|$ greater than r_0^2, where r_0 is a characteristic range of the order 10^{-13} cm. The authors consider a non-local field theory in which the "effective source" $\bar{\Phi}(x)$ generated by a particle density $\Phi(x)$ is given by

$$(1)\qquad \bar{\Phi}(x)=\int F(x-x')\Phi(x')dx'.$$

For details of such a theory see H. MacManus [Proc. Roy. Soc. London. Ser. A. 195, 323–336 (1948); these Rev. 10, 664]. They find conditions on F which are sufficient to ensure that, for any function $\Phi(x)$ having a spread of the order of magnitude r_0 or greater, the spread of $\bar{\Phi}$ differs from that of Φ by distances not greater than r_0. By the "spread" of a function is meant the set of points on which it differs appreciably from zero. The sufficient conditions are

$$(2)\qquad M_n=\int s^n F(s)ds=0,\quad n=0, 1, 2, \cdots.$$

Another sufficient condition is that all derivatives of the Fourier transform of $F(x)$ should be bounded. A theory in

which these conditions are satisfied is macroscopically causal, in the sense that non-causal effects could be observed only inside distances of the order of r_0. Examples are given of form factors satisfying the conditions.

The authors assert that a theory which is macroscopically causal will satisfy the conditions required by W. Pauli [see the preceding review] for the construction of a classical Hamiltonian dynamics. The reviewer considers that the "normal class" of theories, to which the analysis of Pauli applies, is subject to much more severe restrictions than macroscopic causality. *F. J. Dyson* (Princeton, N. J.).

Salam, Abdus, and Matthews, P. T. Fredholm theory of scattering in a given time-dependent field. Physical Rev. (2) **90**, 690–695 (1953).

Consider the scattering of a Dirac electron by a given electromagnetic potential A^e. Let $M(p, q)$ be the scattering matrix between states of momenta q and p. This satisfies the integral equation

$$(1)\quad M(p, q) = \lambda A^e(p-q) - 2\pi\lambda \int A^e(p-k)S_F(k)M(k, q)d_4k.$$

Here λ is the electron charge and S_F is the Feynman propagation kernel for the electron [R. P. Feynman, Physical Rev. (2) **76**, 769–789 (1949); these Rev. **11**, 765]. Now equation (1) can be solved by the standard Fredholm theory [J. Plemelj, Monatsh. Math. Phys. **15**, 93–128 (1904)], which gives the result

$$(2)\quad M(p, q) = S(p, q)/d(\lambda),$$

where $S(p, q)$ and $d(\lambda)$ are integral functions of λ defined by convergent power series expansions. The authors' result is that $S(p, q)$ by itself is the correctly normalized scattering amplitude as defined by Feynman. The denominator $d(\lambda)$, which is the probability amplitude for the vacuum state to remain the vacuum state, is to be simply omitted from (2) in order to obtain correctly normalized scattering probabilities. *F. J. Dyson* (Princeton, N. J.).

Sokolov, A. A. Remarks on the quantum theory of a gravitational field. Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk **1952**, no. 9, 9–20 (1952). (Russian)

The author develops the quantum theory of the linearized field equations of general relativity, verifying that this field is equivalent to an assembly of noninteracting particles with zero mass and spin 2. He shows how a suitable transformation of coordinates will eliminate longitudinally polarized waves, so that there remain just two states of transverse polarization for a particle of given momentum. These results are all well-known [see M. Fierz, Helvetica Phys. Acta **13**, 45–60 (1940); these Rev. **1**, 352], and the author follows the standard methods of deriving them. In the middle of his exposition he addresses some hostile remarks to the "defenders of Machism" and observes that "the reactionary principle of equivalence has no connection with the investigation of the nature of gravitation". *F. J. Dyson.*

Rayski, Jerzy. On a regular field theory. I. Classical. Acta Phys. Polonica **11**, 314–327 (1953).

The author here shows how his classical field theory with non-local interactions [same Acta **11**, 109–130 (1952); these Rev. **14**, 705; and earlier papers there cited] may be derived from a principle of stationary action. From the action principle he deduces in the usual way integral conservation laws for the total charge, energy, momentum, and angular momentum of the system. In this formalism differential conservation laws are not obtained. In an appendix it is proved that, under certain conditions on the form-factor of the non-local interaction, regular solutions of the field equations will always exist. *F. J. Dyson.*

Kawaguchi, Masaaki, and Mugibayashi, Nobumichi. Some consequences of gauge invariance. Progress Theoret. Physics **8**, 212–220 (1952).

Consider any reaction involving elementary particles, in which the incoming and outgoing particles have specified momenta and polarizations. The matrix element for such a reaction will have the form $M = R\chi$ where χ is a product of field operators creating and annihilating the particles in the final and initial states, and R is a numerical tensor whose transformation properties are such as to make M an invariant under Lorentz transformations. If one or more of the particles is a photon, the gauge-invariance of M further restricts the form of R. Therefore, for a given reaction, independently of any knowledge of the details of the interactions, the most general possible form of R can be written down explicitly, as a linear combination of a finite set of covariants. The authors give a simple method of finding the general form of R, and apply the method to various special cases. In particular, they consider the decay of a spin-zero meson into a spin-zero particle plus a photon, and the decay of a spin-one meson into two photons. In both cases there is no allowed form for R, which verifies the fact that these reactions are strictly forbidden in all versions of field theory. *F. J. Dyson* (Princeton, N. J.).

Utiyama, Ryôyû, Sunakawa, Sigenobu, and Imamura, Tsutomu. On the theory of the Green-functions in quantum-electrodynamics. Progress Theoret. Physics **8**, 77–110 (1952).

The purpose of the Green's-function formalism of Schwinger [Proc. Nat. Acad. Sci. U. S. A. **37**, 452–455, 455–459 (1951); these Rev. **13**, 520] is to derive from quantum field theory a set of equations of motion for systems containing a given number of particles. The equations of motion are written in closed form, without expansion in powers of the coupling constant e. However, the equations contain functional derivatives, which in practice prevent us from finding solutions, or even discussing the existence of solutions, except in terms of series expansions.

The authors here develop the Schwinger formalism from the beginning entirely in terms of series expansions. The Green's functions have a simple definition in terms of Feynman diagrams, and the meaning of the whole formalism becomes transparently clear. The equations of Schwinger are derived from the definitions. In the second half of the paper it is shown how the divergences may be removed from the Green's functions by mass and charge renormalization to every power in e, following exactly the method used by the reviewer [Physical Rev. (2) **75**, 1736–1755 (1949); these Rev. **11**, 145] to remove divergences from the S-matrix. *F. J. Dyson* (Princeton, N. J.).

Utiyama, Ryôyû, and Imamura, Tsutomu. Difficulty of divergence of the perturbation method in the quantum field theory. Progress Theoret. Physics **9**, 431–454 (1953).

The authors prove the divergence of the series expansion of the S-matrix in powers of the coupling-constant g for a neutral scalar meson field theory with a non-linear term $g\phi^3$ in the Lagrangian. The result, and the method of proof, is

essentially the same as that used independently by Thirring [Helvetica Phys. Acta **26**, 33–52 (1953); these Rev. **14**, 708] and by Hurst [Proc. Cambridge Philos. Soc. **48**, 625–639 (1952); these Rev. **14**, 607]. However, the present authors calculate with an S-matrix cut-off arbitrarily at high momenta, and they do not discuss adequately what happens when the cut-off is removed and the resulting divergence absorbed by a mass renormalization. Of the three published treatments of the problem, only Thirring's is complete in this respect. *F. J. Dyson* (Princeton, N. J.).

Edwards, S. F. A nonperturbation approach to quantum electrodynamics. Physical Rev. (2) **90**, 284–291 (1953).

The author considers the problem of solving the system of coupled non-linear integro-differential equations derived by J. Schwinger [Proc. Nat. Acad. Sci. U. S. A. **37**, 452–455, 455–459 (1951); these Rev. **13**, 520] as a starting-point for quantum electrodynamics. He hopes to be able to solve these equations without using a perturbation expansion in powers of the fine-structure constant α. In this paper he takes one of Schwinger's equations and linearizes it by dropping the non-linear terms. The approximation is similar to one that has been frequently made, and is called the "ladder" approximation, in the theory of the two-particle integral equation of Bethe and Salpeter [Physical Rev. (2) **84**, 1232–1242 (1951); these Rev. **14**, 707]. It is known to be a bad approximation in many cases. The linearized equation, after some further approximations, takes the form

$$(1)\qquad f(j^2)=\frac{-2i\lambda}{\pi^3}\int[(l^2+m^2)(j-l)^2]^{-1}f(l^2)dl,$$

where j and l are variable 4-vectors and m and λ constants. This has the solution

$$(2)\qquad f(j^2)=\frac{1}{1-\lambda}\int_0^1\left|\frac{m^2(1-z)}{z(j^2z+m^2)}\right|^\lambda dz.$$

Using this solution, various physical quantities are calculated in closed form without power series expansion in $\alpha=8\pi\lambda$. *F. J. Dyson* (Princeton, N. J.).

Takahashi, Yasushi, and Umezawa, Hiroomi. The general theory of the interaction representation. I. The local field. Progress Theoret. Physics **9**, 14–32 (1953).

This paper is a systematization of the methods for writing down the commutation relations and the Schrödinger equation, which constitute the "interaction representation" description of a system of interacting quantized fields. The results are not new but they are derived here in a simple, direct and general way. For example, let a set of free field operators $Q_\alpha(x)$ satisfy the linear field equations

$$(1)\qquad \Lambda_{\alpha\beta}Q_\beta(x)=0.$$

Then the covariant commutation rule for the Q_α is

$$(2)\qquad [Q_\alpha(x),Q_\beta(x')]_\pm=iR_{\alpha\beta}\Delta(x-x'),$$

where $R_{\alpha\beta}$ is a differential operator defined by

$$(3)\qquad \Lambda_{\alpha\beta}R_{\beta\gamma}=\delta_{\alpha\gamma}(\Box-\kappa^2),$$

and κ is the characteristic rest-mass of the field. The authors derive from (2) in a very simple way the known commutation relations for spin-1 and spin-3/2 fields. They then discuss in a similar way the construction of the interaction Hamiltonian and of the energy-momentum tensor for a system of interacting fields. *F. J. Dyson.*

Kamefuchi, Susumu, and Umezawa, Hiroomi. On the structure of the interactions of the elementary particles. III. On the renormalizable field theory. Progress Theoret. Physics **8**, 579–598 (1952).

This paper is a sequel to two others [Sakata, Umezawa, and Kamefuchi, same journal **7**, 377–390 (1952); Umezawa, ibid. **7**, 551–562 (1952); these Rev. **14**, 608, 709]. The authors summarize the possible variety of renormalizable field theories, i.e., theories containing only interactions of the first kind. They then explain the two methods, due to Salam [Physical Rev. (2) **84**, 426–431 (1951); these Rev. **13**, 608] and Ward [ibid. **84**, 897–901 (1951)] respectively, which may be used for calculating with such theories. Either method results in a complete elimination of divergences from the calculation. However, the authors show that the two methods may in some circumstances give different finite expressions for the same observable quantity. Whether this difference results from a different definition of the "renormalized coupling constant" in the two cases, or whether there is a real difference in the predictions of the two methods, the authors do not investigate. They give finally a verification that the charge-renormalization in every renormalizable theory is the same for all charged particles and is independent of their mass and spin. This independence results directly from the gauge-invariance of the theory.

F. J. Dyson (Princeton, N. J.).

Kamefuchi, Susumu, and Umezawa, Hiroomi. On the structure of the interaction of the elementary particles. IV. On the interaction of the second kind. Progress Theoret. Physics **9**, 529–549 (1953).

In this paper the authors entirely change the point of view of their preceding paper [see the paper reviewed above] and propose a method of making quantum field theory divergence-free without any subtraction or renormalization procedure. Their idea is to add to the Lagrangian of the theory terms which are quadratic in the field variables and involve derivatives higher than the first. The Green's functions of the free fields then are less singular than the Green's functions of the usual theory, because the higher derivatives introduce high powers of the momentum into the denominators of the Fourier transforms of the Green's functions. So the Feynman integrals taken with the modified Green's functions will converge. And by taking sufficiently high derivatives in the added terms, every field theory becomes convergent, even when it contains interactions of the second kind.

To the reviewer it seems that these proposals are in essence identical with those made several years ago by Bopp, Feynman, and others. For references, and for a thorough discussion of the objections to these proposals, see Pais and Uhlenbeck [Physical Rev. (2) **79**, 145–165 (1950); these Rev. **12**, 227]. *F. J. Dyson* (Princeton, N. J.).

Wildermuth, Karl. Die Grenzen der Quantentheorie der Wellenfelder. Z. Naturforschung **8a**, 105–116 (1953).

This paper is motivated by the conjecture of Heisenberg—previously entertained by Eddington and de Broglie among others—that particles with spin greater than one-half, such as light-quanta and mesons, are compounds of spinor particles. Interactions between compound particles are hence integrations of the interactions between their constituents. A problem, drastically simplified compared with the "real" world, is studied in the hope of gaining insight into the limits of the validity of current field theories if they are regarded as approximations to a theory of the above type.

The author concludes that current meson theories cannot be expected to give sensible results for particles with de Broglie wavelength less than 10^{-13} cm., and therefore they are useless. *A. J. Coleman* (Toronto, Ont.).

Waldmann, Ludwig. Die Erhaltungsgrössen der klassischen Feldmechanik. Z. Naturforschung **8a**, 417–428 (1953).

The classical theory of Bopp is expressed as an action principle and the motion of a particle in the absence of an external field is examined up to terms in the Lagrangian involving second derivatives. In this case the motion may be uniform circular, so that the particle has a natural classical intrinsic angular momentum. The electromagnetic contribution to its rest-mass depends on the form-factor introduced, and when this latter is prescribed the arbitrary constants can be chosen so that for a given radius the total energy is a minimum. Detailed results are given for a square-well form-factor and different ratios of electromagnetic to non-electromagnetic mass. *H. C. Corben.*

Wildermuth, K. Strenge Lösungen von Mehrkörperproblemen. Acta Physica Austriaca **7**, 299–310 (1953).

Two three-body problems are solved explicitly in order to provide exact solutions for comparison with the results of approximate methods. In each case, particle 3 at the origin is infinitely heavy, particle 2 interacts with it via a delta function and particle 1 is scattered. In the first case, 1 interacts with 2 but not with 3. In the second case, 1 and 2 are identical particles interacting with 3 and with each other. *A. J. Coleman* (Toronto, Ont.).

Baumann, Kurt. Zur Definition der relativistischen Teilchen-Antiteilchen-Wellenfunktion. Acta Physica Austriaca **7**, 98–101 (1953).

The author considers the relativistic wave-function describing a bound state of a particle with its anti-particle, e.g., a positronium atom. This wave-function may be constructed in two alternative ways. One is the method of Bethe and Salpeter [Physical Rev. (2) **84**, 1232–1242 (1951); these Rev. **14**, 707] who describe the positron by the ordinary Dirac wave-function of the negative energy electron state to which the positron corresponds. The other is the method of R. Karplus and A. Klein [ibid. **87**, 848–858 (1952)] who describe the positron by a "charge conjugate" positive-energy Dirac wave-function. The author asserts that the Karplus-Klein wave-function gives a correct description of a bound state, but that the use of the Bethe-Salpeter wave-function is unjustified. His arguments are too condensed to be intelligible to the reviewer. The reviewer has no doubt that the published treatment of the positronium atom by Karplus and Klein is correct, but he believes that the method of Bethe and Salpeter is precisely equivalent to it and would lead to identical results. *F. J. Dyson* (Princeton, N. J.).

Donnert, Hermann. Ableitung mathematisch äquivalenter Tensorgleichungen aus den Diracschen Spinorgleichungen für das Wellenfeld von Elementarteilchen nicht verschwindender Masse und beliebiger ganzer Spinquantenzahl. Acta Physica Austriaca **7**, 181–197 (1953).

Im Bestreben die Physik der verschiedenen Partikel auf Grund von Verwandlungen eines einheitlichen Spinorfeldes zu verstehen und darzustellen, sind Dirac's Ergebnisse über relativistisch invariante Feldgleichungen im Falle von Wellenfeldern von Elementarteilchen beliebiger ganz- oder halbzahliger Spinquantenzahlen von grosser Bedeutung geworden. Verf. stellt sich die Aufgabe diese Diracschen Spinorgleichungen im Falle von Elementarteilchen nicht verschwindender Masse und beliebiger ganzer Spinquantenzahl in äquivalente Tensorgleichungen zu verwandeln. Die Transformationsgruppe $\mathfrak{U}$ der Spinoren ist die unimodulare Gruppe vom Rang 2. Sie bildet eine zweideutige Darstellung der eigentlichen Lorentzgruppe L^+, wenn man zur Zuordnung zwischen Spinoren und Tensoren folgende Matrizen verwendet

$$\sigma^{1\dot\mu\nu}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\quad \sigma^{2\dot\mu\nu}=\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix},$$
$$\sigma^{3\dot\mu\nu}=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\quad \sigma^{4\dot\mu\nu}=\begin{bmatrix}-i & 0\\ 0 & -i\end{bmatrix}.$$

Dann ergibt sich z.B. der Zusammenhang

$$A^n=\tfrac{1}{2}\sigma^{n\dot\mu\nu}a_{\dot\mu\nu}=\tfrac{1}{2}\sigma^n{}_{\dot\mu\nu}a^{\dot\mu\nu}=\tfrac{1}{2}\sigma^{n\dot\mu}{}_\nu a_{\dot\mu}{}^\nu=\tfrac{1}{2}\sigma^n{}_{\dot\mu}{}^\nu a^{\dot\mu}{}_\nu,$$

zwischen den assoziierten Spinoren $a_{\dot\mu\nu}$, $a^{\dot\mu\nu}$, $a_{\dot\mu}{}^\nu$, $a^{\dot\mu}{}_\nu$ und dem kontravarianten Vektor A^n und ähnlich für kovariante Vektoren A_m. Die Umkehrung dieser Formeln hat die Gestalt $a^{\dot\mu\nu}=\rho^{n\dot\mu\nu}A_n=\rho_n{}^{\dot\mu\nu}A^n$ und ähnlich für $a_{\dot\mu\nu}$, $a^{\dot\mu}{}_\nu$ und $a_{\dot\mu}{}^\nu$. Zwischen den Grössen σ und ρ bestehen eine Reihe wichtiger Beziehungen, die Verf. für die weitere Untersuchung systematisch zusammenstellt. Auch die Differentialoperatoren des Tensorkalküls lassen sich in Spinorform bringen, indem die $\partial_m=\partial/\partial x^m$ und $\partial^n=\partial/\partial x_n$ formal wie Weltvektoren behandelt werden. Z.B. ergibt sich für den Schrödinger-Gordon-Operator:

$$\Box=\partial^n\partial_n=\tfrac{1}{2}\partial_{\dot\mu}{}^\nu\partial^{\dot\mu}{}_\nu,\quad \partial^{\dot\mu\nu}\partial_{\dot\mu\lambda}=-\delta_\lambda{}^\nu\Box,\quad \partial^{\dot\mu\nu}\partial_{\dot\kappa\nu}=-\delta_{\dot\kappa}{}^{\dot\mu}\Box.$$

Ein Spinor heisst vollsymmetrisch, wenn bezüglich jedes Paares gleichwertiger Indizes Symmetrie herrscht. Jedem Spinor vom Rang $2s\equiv 0$ (2) mit genau s unpunktierten und s punktierten Indizes kann ein Tensor der Stufe s zugeordnet werden und umgekehrt. Ist die Zahl der unpunktierten und punktierten Indizes ungleich, so kann man durch Multiplikation mit den metrischen Spinoren

$$\epsilon^{\dot\mu\dot\nu}=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix},\quad \epsilon^{\mu\nu}=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix},$$
$$\epsilon_{\dot\mu\dot\nu}=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix},\quad \epsilon_{\mu\nu}=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$$

diesen Übelstand ausgleichen und hernach wiederum die entsprechenden Tensoren zuordnen. Ist der Spinor vollsymmetrisch, so ist der zugeordnete Tensor symmetrisch in jedem Indexpaar und spurfrei und umgekehrt. Mit dem so entwickelten Kalkül behandelt Verf. nunmehr die Diracschen Feldgleichungen im Falle ganzzahligen Maximalspins $2s\equiv 0$ (2):

$$\partial^{\dot k_0 r_1}a^{\dot k_1\dot k_2\cdots\dot k_s}_{r_1r_2\cdots r_s}=i\kappa b^{\dot k_0\dot k_1\dot k_2\cdots\dot k_s}_{r_2\cdots r_s},\quad \partial_{\dot k_0 r_1}b^{\dot k_0\dot k_1\dot k_2\cdots\dot k_s}_{r_2\cdots r_s}=i\kappa a^{\dot k_1\dot k_2\cdots\dot k_s}_{r_1r_2\cdots r_s},$$
$$\kappa=mc/\hbar\neq 0,\quad (m\neq 0),$$

und gewinnt die äquivalenten Tensorgleichungen:

$$F^{r_0r_1r_2\cdots r_s}=\partial^{r_0}A^{r_1r_2\cdots r_s}\partial^{r_1}A^{r_0r_2\cdots r_s},\quad \partial_{r_0}F^{r_0r_1r_2\cdots r_s}=\kappa^2A^{r_1r_2\cdots r_s}.$$

Der Tensor $F^{r_0r_1\cdots r_s}$ ist spurfrei in allen Indexpaaren aus $0, 1, 2, \cdots, s$, symmetrisch in allen Indexpaaren aus $2, 3, \cdots, s$ und schiefsymmetrisch in r_0, r_1. Die Vektordivergenz von A verschwindet. Verf. zeigt schliesslich die Äquivalenz der erhaltenen Tensorgleichungen, indem er aus ihnen rückwärts wiederum Dirac's Spinorgleichungen herleitet. *M. Pinl* (Dacca).

Gurevič, A. V. On the classical theory of extended particles. Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk **1952**, no. 8, 105–109 (1952). (Russian)

This paper consists of some purely formal deductions from Blohincev's theory of extended particles [Vestnik Moskov. Univ. **1948**, no. 1, 83–91; these Rev. **10**, 345]. The author chooses for the "smearing function", which gives the shape of the extended charge, the special form

$$\Delta(x)=\int\left[\frac{-k_0^2}{k^2-k_0^2}\right]\exp(ik\cdot x)d_4k,$$

where k_0 is a constant defining the reciprocal "radius" of the charge. With this choice, the electromagnetic potentials A_μ are given in terms of the source-density j_μ by the fourth-order wave-equation

$$\Box(1-k_0^{-2}\Box)A_\mu=-4\pi j_\mu.$$

The potentials of the Bopp-Podolsky electrodynamics [F. Bopp, Ann. Physik (5) **38**, 345–384 (1940); these Rev. **2**, 336] satisfy the same equation. However, the author points out that the two theories are not equivalent, and in particular the finite self-energies of a charged particle are different in the two theories. *F. J. Dyson.*

Jauho, Pekka. On the commutation relations and vacuum expectation values in the quantum theory of fields. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **127**, 8 pp. (1952).

Assuming Lorentz covariance, the author finds the most general form possible for the commutators and vacuum expectation values in neutral scalar field theory. He concludes that no amount of trickery with the commutators or the definition of the vacuum can eliminate divergences from the theory. *A. J. Coleman* (Toronto, Ont.).

Jordan, Hermann L. Begrenzung der Lokalisierbarkeit von Wechselwirkungen in der Quantentheorie der Elementarteilchen und Felder. Z. Naturforschung **8a**, 341–352 (1953).

Consequences of introducing a minimum length into the quantum theory of fields are investigated and the connection with the theories of Born, Bopp, and others is examined. Replacing $\Psi(x)$ in Dirac's equation by $G^*\Psi=\int G(x-x')\Psi(x')dx'$, and similarly for A, one may introduce the minimum length by appropriate choice of G, and in terms of the notation $G^*\Psi$ many results of the theory become formally similar to those of the usual local theory. *H. C. Corben* (Pittsburgh, Pa.).

Ma, S. T. Bound states and the interaction representation. Physical Rev. (2) **87**, 652–655 (1952).

The properties of the wave matrix W and the scattering matrix S of a Hamiltonian containing a free particle term and an interaction term are discussed in the time-independent formalism and in the interaction representation. The possible existence of bound states for the complete Hamiltonian is taken into account and its consequences for the unitarity of W are exhibited. The important distinction is made between strict equalities and "conditional equalities"; the latter contain limiting processes to be performed with the help of smoothing factors.
L. Van Hove (Princeton, N. J.).

Ma, S. T. Power-series expansion of the unitary operator $U(t, t_0)$. Physical Rev. (2) **91**, 392 (1953).

The author considers a one-dimensional particle in a potential $-2b\delta(x)$ and shows that the unitary operator of the motion over a finite time interval has a convergent power expansion in b, despite the existence of a bound state for $b>0$. *L. Van Hove* (Princeton, N. J.).

Güttinger, Werner. Quantum field theory in the light of distribution analysis. Physical Rev. (2) **89**, 1004–1019 (1953).

The author develops a treatment of S-matrix elements in the usual perturbation theory-interaction representation approach in which the divergent integrals arising from closed loop processes are interpreted as distributions (in the sense of Schwartz) applied to functions. This involves in particular a rigorous and succinct treatment of the singular propagation functions of field theory via distributions and especially by means of a generalization of the notion of principal value. While the S-matrix results are in general somewhat ambiguous, by virtue of the lack of unique quotients for distributions, the degree of the ambiguity is sharply defined, and in a number of basic cases can be eliminated by covariance considerations. It is indicated that in these cases the distribution-theoretic approach gives results equivalent to those of Feynman and Schwinger.
I. E. Segal (Chicago, Ill.).

Thermodynamics, Statistical Mechanics

Fényes, Imre. Die Anwendung der mathematischen Prinzipien der Mechanik in der Thermodynamik. Z. Physik **132**, 140–145 (1952).

A formal parallel is drawn between the principles of classical Hamiltonian dynamics and those of phenomenological thermodynamics, particularly in the irreversible case and with certain extensions. Some general conclusions are drawn and suggested; everything is on the formal rather than the detailed structural level, and statistics plays, if anything, a subordinate part. *B. O. Koopman.*

Scheidegger, A. E., and Krotkov, R. V. Relativistic statistical thermodynamics. Physical Rev. (2) **89**, 1096–1100 (1953).

The object of this paper is the formulation of statistical mechanics consistent with quantum mechanics and relativity theory. This is achieved by methods simpler than those used by P. G. Bergman [Physical Rev. (2) **84**, 1026–1033 (1951); these Rev. **14**, 231] and similar to those used in ordinary statistical mechanics. *L. Infeld* (Warsaw).

Dutta, M. An essentially statistical approach to the thermodynamic problem. Proc. Nat. Inst. Sci. India **19**, 109–126 (1953).

The behavior of a thermodynamical system is investigated by purely statistical methods, without the introduction of any assumption concerning the structure or internal mechanism of the system. For a system of given volume in a given environment, there is formulated the probability that the system will be in a state of given energy and matter. Observed states are determined by maximizing this probability. The usual formulas for fluctuations are developed, and such quantities as temperature and entropy are identified as certain functions associated with the equations of continuity. *C. C. Torrance* (Monterey, Calif.).

Hall, Newman A. Nonequilibrium thermodynamics. J. Appl. Phys. **24**, 819–825 (1953).

An expository account of recent theories of irreversible processes. *C. C. Torrance* (Monterey, Calif.).

Halatnikov, I. On a method of computation of a statistical sum. Doklady Akad. Nauk SSSR (N.S.) **87**, 539–542 (1952). (Russian)

Consider a quantum statistical-mechanical system with the Hamiltonian

$$(1) \qquad H=(p^2/2m)+U(r).$$

The partition function is given by

$$(2) \qquad Z=\mathrm{Spur}\exp[-\beta H], \quad \beta=[kT]^{-1}.$$

R. Feynman [Physical Rev. (2) **84**, 108–128 (1951); these Rev. **13**, 410] has developed a general calculus for evaluating $\exp[A+B]$, where A and B are non-commuting operators. The author applies Feynman's calculus to the evaluation of Z. Expressing Z as a power series in Planck's constant h,

$$(3) \qquad Z=Z_0+h^2Z_2+h^4Z_4+\cdots,$$

Z_0 is the classical partition function and $Z_2, Z_4, \cdots$ are quantum-mechanical corrections. Explicit forms of Z_2 and Z_4 are obtained very quickly by Feynman's method, for example,

$$(4) \qquad Z_2=-(1/24)(2m/\pi\beta)^{1/2}\int|\beta \operatorname{grad} U|^2\exp(-\beta U)dr.$$

These results were obtained more laboriously by J. Kirkwood [ibid. **44**, 31–37 (1933)]. *F. J. Dyson.*

Fišer, I. Z. A new derivation and physical interpretation of the equations of Bogolyubov for equilibrium functions of a distribution. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **21**, 1109–1112 (1951). (Russian)

Bogolyubov derived a set of integro-differential equations for the cluster distribution functions by expanding the Gibbs distribution [Problems of dynamical theory in statistical physics, Gostehizdat, Moscow-Leningrad, 1946; these Rev. **13**, 196]. The starting point of the present derivation is the diffusion equation. On writing the force as an average over the distribution of pairs and introducing the potential energy, one obtains the first Bogolyubov equation involving clusters of one and two molecules. The two equations for the higher clusters are obtained by total induction. *L. Tisza* (Cambridge, Mass.).

***Landau, L., and Lifšic, E. Statističeskaya fizika (klassičeskaya i kvantovaya). [Statistical physics (classical and quantum)].** Gosudarstv. Isdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951. 479 pp. 14.75 rubles.

The present volume (the 4th in a series on Theoretical Physics) contains the revised and considerably enlarged material of an earlier book of the authors published under the same title [Oxford, 1938]. Whereas the first edition contained a unified theory of thermodynamics and classical statistical mechanics following to a large extent the ideas of Gibbs, the present volume includes also quantum statistics. Frequently, a classical result is immediately followed by the quantum statistical reformulation. Elegant but approximate mathematical methods make it possible to cover a large number of applications in comparatively small space. The problems connected with the macroscopic electric and magnetic properties of matter are left for another volume. A selection of topics follows: Gibbs distributions leading to thermodynamics, increase of entropy, a fact asserted to present a paradox, thermodynamic transformations by means of Jacobians. Thermodynamic inequalities, Nernst theorem, rotating bodies, relativistic generalizations, perturbation methods in the partition sum, various types of ideal gases, solids, phonons, superfluidity on the basis of phonon and Bose-type energy spectrum of quantum liquid. (It is emphasized that the Bose-type spectrum may not be necessarily connected with the statistics of the constituent particles.) The two-fluid model is not treated. Negative temperatures. Classical and quantum non-ideal gases, second virial coefficients, Coulomb interaction, Debye-Hückel theory. Phase equilibrium, critical points. Solutions, types of equilibrium curves. Chemical equilibrium. Properties of matter at high temperatures and densities with astrophysical applications. Fluctuations, Gaussian distributions, Poisson formula, correlation of fluctuations, fluctuations at the critical points (the theory is based on the "capillary" effect of Rocard rather than on Ornstein-Zernicke's correlation effects). Radial distribution functions in ideal quantum gases. Correlations in time. Onsager type thermodynamics of irreversible phenomena, dissipation function. The role of symmetry in solids, crystal classes. Phase transitions of the second kind are treated only on the basis of Landau's expansion of the Gibbs function around the critical point. Ehrenfest type discontinuities are found for the specific heat rather than singularities as in more rigorous theories. The latter cast doubt on the validity of Landau's expansion, since the Gibbs function is presumably singular at the critical point [cf., e.g., Smoluchowski, Mayer, and Weyl, Phase transformations in solids, Wiley, New York, 1951, p. 1]. Surface effects in fluids and crystals. The average length of long-chain molecules. There are a large number of solved problems. Bibliography is almost completely absent. *L. Tisza* (Cambridge, Mass.).

Bergmann, Peter G., and Thomson, Alice C. Generalized statistical mechanics and the Onsager relations. Physical Rev. (2) **91**, 180–184 (1953).

The Onsager symmetry relations for irreversible processes are derived for thermally isolated systems differing slightly from a canonical distribution, use being made of the "generalized statistical mechanics" discussed by P. G. Bergmann in a previous paper [Physical Rev. (2) **84**, 1026–1033 (1951); these Rev. **14**, 231]. *L. Van Hove.*

Glauberman, A. E. On the derivation of the equations for the equilibrium functions of a distribution of molecules. Doklady Akad. Nauk SSSR (N.S.) **89**, 659–662 (1953). (Russian)

The distribution functions of positions and velocities for groups of particles belonging to a system of interacting particles are defined and the continuity equation they fulfill is given in terms of the forces. Assuming that the equilibrium distribution is the product of a distribution function for positions and a distribution function for velocities, both time independent, one obtains the Maxwell distribution for velocities and the familiar integro-differential equation for the equilibrium distribution of positions [N. N. Bogolyubov, Problems of dynamical theory in statistical physics, Gostehizdat, 1946; these Rev. **13**, 196; M. Born and H. Green, A general kinetic theory of liquids, Cambridge, 1949; these Rev. **12**, 230]. Justification of the above-mentioned assumption is said to fall beyond the scope of the note under review. *L. Van Hove* (Princeton, N. J.).

Osborne, M. F. M. Number theory and the magnetic properties of an electron gas. Physical Rev. (2) **88**, 438–451 (1952).

The subject of this paper is the magnetization produced by an external magnetic field in a gas of completely free electrons moving in a finite container. Classically the magnetization is exactly zero, as shown by N. Bohr [Dissertation, Copenhagen, 1911], because the large negative contribution of the electrons which do not collide with the walls is just cancelled by the positive contribution from those which do collide. Quantum-mechanically the cancellation is no longer exact, and a residual magnetization exists of the order of one millionth part of the large cancelling terms. The theory of the residual quantum-mechanical magnetization has been treated by many authors, with wildly inconsistent results. In the present paper the author explains the earlier inconsistencies as caused by mathematical approximations which were too crude for a calculation in which so high a degree of cancellation is involved. He here presents a calculation in which all the approximations are controlled, if not with mathematical rigor, at least with sufficient care so that physicists may accept the results as reliable. The results are that the magnetization is always very small and contains a term varying periodically with the field strength [de Haas-van Alphen effect].

The calculation reduces to finding the number of quantum states of an electron in the magnetic field with total energy less than a given limit E_0. A quantum state is defined by three quantum numbers which take integer values, and the energy of the state is a known algebraic function of the quantum numbers. So the number of states is equal to the number of points (x, y, z) with integer coordinates inside a surface $F(x, y, z)=E_0$. This number has to be calculated with such accuracy that terms proportional to the surface area are not negligible. The author takes from D. G. Kendall [Quart. J. Math., Oxford Ser. **19**, 1–26 (1948); these Rev. **9**, 570] an asymptotic formula for the number of lattice points inside a surface of general shape. The deeper number-theoretical problems, concerned with the order of magnitude of the error term in the asymptotic formula, are here not relevant. The calculations are carried out in detail for the special case of a cylindrical container with axis parallel to the magnetic field. As the results are in agreement with those calculated by M. C. Steele for a cubical container [see the following review], they are probably valid for a container of any reasonable shape. *F. J. Dyson.*

Steele, M. C. Application of the theory of numbers to the magnetic properties of a free electron gas. Physical Rev. (2) **88**, 451–464 (1952).

This is a companion paper to that of Osborne [see the preceding review]. The author deals with the same problem as Osborne, only assuming a cubical instead of a cylindrical container; and he carries the calculations further, so that many terms which were estimated by Osborne only for order of magnitude are here given with precise numerical coefficients. To obtain these more precise evaluations of the lattice-point sums, he used the "method of critical points" of J. G. van der Corput [Nederl. Akad. Wetensch., Proc. **51**, 650–658 (1948); these Rev. **10**, 112]. This is a generalization of the well-known "method of steepest descents". In the last section the author calculates the effect on his results of the electron spin, which up to that point had been ignored. *F. J. Dyson* (Princeton, N. J.).

Zwanzig, Robert W. The quantum-mechanical equation of state. J. Chem. Phys. **18**, 1412–1413 (1950).

Born and Green [Proc. Roy. Soc. London. Ser. A. **191**, 168–181 (1947); these Rev. **9**, 402] and H. S. Green [Physica **15**, 882–890 (1949); these Rev. **11**, 634] claim that the pressure of a quantum gas as computed from the partition function is not always the same as that computed by the virial theorem, although the two are identical in a classical system. Many authors, for example J. de Boer [Nuovo Cimento (9) **6**, supplemento, 199–207 (1949); these Rev. **12**, 575], Riddell and Uhlenbeck [J. Chem. Phys. **18**, 1066–1069 (1950); these Rev. **12**, 575], Yvon [Nuovo Cimento **6**, supplemento, 187–197 (1949); these Rev. **12**, 576] have questioned this claim. The main mathematical point of Born and Green is that the trace of commutators of two infinite matrices might not vanish even though that of two finite matrices does. Such commutators appear in some of the above mentioned papers. The paper reviewed here refutes the Born-Green claim without employing the questioned commutators in the analysis. *E. W. Montroll.*

Frisch, Harry L. An equipartition principle of generalized canonical ensembles. Physical Rev. (2) **91**, 791–793 (1953).

The standard derivation of the equipartition principle is carried out in the framework of the "generalized statistical mechanics" proposed by P. G. Bergmann [Physical Rev. (2) **84**, 1026–1033 (1951); these Rev. **14**, 231] and two applications are given. *L. Van Hove.*

ter Haar, D. Statistics of the three-dimensional ferromagnet. II. Comparison of various approximation methods. Physica **18**, 836–840 (1952).

[For part I see Martin and ter Haar, Physica **18**, 569–581 (1952); these Rev. **14**, 522.] The author lists seven power-series expansions for the largest characteristic root of the Kramers-Wannier matrix for the three-dimensional Ising model including one he derived himself. The longest of these has five terms, i.e., has 0, 2, 4, 6, and 8th powers of $K=J/kT$. *F. J. Murray* (New York, N. Y.).

Harasima, Akira. Statistical mechanics of surface tension. J. Phys. Soc. Japan **8**, 343–347 (1953).

Ono, Syû. Statistical mechanics of phase transition. Progress Theoret. Physics **8**, 1–12 (1952).

In connection with the objection raised by Katsura and Fujita to the treatment by Mayer of phase transition, the author considers two systems, one of which is a cubical lattice crystal with a two-state possibility and the other a generalization in which a continuous range of energy is permitted. The objection to Mayer's work is that he considers an infinite series whose coefficients are limits of functions of V while in the precise situation, the limit as V approaches infinity, should be taken after the infinite series has been summed. In the crystal lattice with two-state systems the author repeats the discussion of Montroll for a crystal with finite rectangular cross-section in which Montroll has shown that there is no singularity. Associated with this finite cross-section is the Kramers-Wannier matrix whose largest characteristic root readily yields the free energy. The author considers what happens in the case in which this cross-section approaches infinity. Using an assumption which to the reviewer seems to be a combination of physical and mathematical reasoning the author indicates that there are two cases, in one of which Mayer's procedure

is justified; in the other, the objection may hold. An analogous discussion is given in the continuous-state situation. Equation 1 in this paper does not seem to the reviewer to be equivalent to the usual equation given in textbooks on statistical mechanics. *F. J. Murray.*

Kurata, Michio, Kikuchi, Ryoichi, and Watari, Tatsuro. A theory of cooperative phenomena. III. Detailed discussions of the cluster variation method. J. Chem. Phys. 21, 434–448 (1953).

The "cluster variation method" of this paper is a new approach to certain earlier procedures of Kikuchi [Physical Rev. (2) **81**, 988–1003 (1951); these Rev. **12**, 788] and consists in the following. Consider a two-dimensional lattice of points on each of which a spin variable σ having values 0 or 1 is located. A possibility is obtained by assigning values to each σ. In general it is desired to know the number of possibilities when the number of adjacent pairs of points on which σ has different values is known. Now consider a configuration consisting of a relatively small number of points in the lattice such as the points in a small square or lattice. One enumerates the possibilities for such a configuration and assigns a frequency of occurrence to these possible small configurations consistent with the restriction on the number of adjacent pairs and other restrictions if desired. These small configurations are then used to cover the lattice in agreement with the specified frequencies with possible overlap between adjacent configurations. The small configurations and the overlapping covering must be such that every bond between neighbors is in some small configuration. The covering with small overlapping configurations will correspond to a possibility only if the σ values assigned to a lattice point on which overlap occurs are the same for all the small configurations which overlap. The fraction of coverings which correspond to possibilities are estimated by approximate methods. The author shows that Bethe's approximation is obtainable by configurations consisting of just pairs of points and overlap is estimated on a probability basis. An approximation due to Kramers and Wannier is obtained from overlapping squares. Other approximations in the plane and three-dimensional cases are discussed. The author compares the results term by term with a high temperature expansion of the partition function in powers of $k=\tanh(\epsilon/kT)$ which is obtained by counting closed polygons. The author studies the polygons whose effect is inaccurately given in each approximation and gives rules for improving a given approximation. He also describes a "dendritic" lattice whose partition function is given by the Bethe approximation. *F. J. Murray.*

Newell, Gordon F., and Montroll, Elliott W. On the theory of the Ising model of ferromagnetism. Rev. Modern Physics 25, 353–389 (1953).

This excellent article is a "review of the work done on the Ising problem (and its equivalents) since the appearance of the comprehensive review of order-disorder phenomenon by Nix and Shockley". Experience with the two-dimensional problem, which has been solved by Onsager, shows that approximations do not accurately describe the critical phenomena and consequently the emphasis is on "exact analytical expressions" for the thermodynamical quantities. Section 1 is introductory. Section 2 gives a modernized version of the Kramers-Wannier matrix formulation and two "combinational" formulations, one of which is the "polygon" process of van der Waerden, the other involves the number of nearest neighbors. In Section 3, duality theorems are established based on the two combinational expressions, and a matrix discussion for the existence of long range order is given. Section 4 discusses the exact solution of the two-dimensional rectangular lattice by two methods. One of these is the Kac and Ward determinant process, the other is a modern version of the original Onsager solution. This version is based on an unpublished suggestion of van der Waerden. The original K-W matrix is used to develop a Lie algebra and this in turn is subject to a Fourier transformation. In the Onsager case, the Lie algebra contains only $3n-1$ linearly independent elements (n is the number of rows in the lattice) but in the three-dimensional problem or in the two-dimensional problem with magnetization a proportionally much larger Lie algebra appears. The explicit solution is used to discuss thermodynamic quantities and a summary of the results on other planar lattices is given. Section 6 discusses ferromagnetism and spin correlations. Section 7 summarizes the various approximate results known for the three-dimensional lattice. Section 8 discusses the spherical model of Berlin and Kac. Appendix 1 gives the relation between the Ising model and the order-disorder problem for a binary alloy and the problem of condensation. Appendix 2 discusses the one-dimensional Ising model. Appendix 4 gives the star triangle transformation for certain planar lattices. Appendices 3 and 5 are expository.

F. J. Murray (New York, N. Y.).

Van Hove, Léon. The occurrence of singularities in the elastic frequency distribution of a crystal. Physical Rev. (2) 89, 1189–1193 (1953).

This paper discusses the application of Morse's Theorem [Trans. Amer. Math. Soc. **27**, 345–396 (1925)] to two functions $\nu(q)$ and $g(\nu)$, the latter being the elastic frequency distribution function for a crystal. In two dimensions q is a point in the plane and $\nu(q)$ is a characteristic root of a matrix whose coefficients have two vector periods and thus $\nu(q)$ can be considered to be defined on a torus. The three-dimensional situation is analogous. $g(\nu)$ is related to $\nu(q)$ by an integral expression involving $(\sum_{\alpha=1}^{l}\nu_\alpha(q)^2)^{-1/2}$ where ν_α refers to the partial of ν with respect to x_α and $l=2$ or 3 is the dimensionality of the crystal considered. Thus points where $\nu_\alpha=0$, $\alpha=1,\cdots,l$, may introduce singularities into $g(\nu)$ provided the integration does not combine a number of these with a cancelling effect. The theorem of Morse specifies the minimum number of singular points with the various possible indices. While a stationary point is a singular point of the Morse type, the latter also includes points at which a tangent plane does not exist. The application of Morse's Theorem to the torus or its three-dimensional equivalent must therefore take these possibilities into account and this is done in the present paper. Confirming the work of Montroll and Smollett, two logarithmically infinite peaks are found for the two-dimensional distribution function. In the three-dimensional case $g(\nu)$ is continuous but its derivative has infinite discontinuities, at least three.

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